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Distributed Space-Frequency Coding over Amplify-and-Forward Relay Channels

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***Abstract*— In this paper, the design of distributed space-frequency codes (DSFCs) for wireless relay networks employing the amplify-and-forward (AAF) protocol is considered. The term distributed comes from the fact that the space-frequency code is distributed among randomly located relay nodes. DSFCs are designed to achieve the multi-path (frequency) and cooperative diversities of the wireless relay channels. We derive sufficient conditions for the proposed code design to achieve full diversity based on minimizing the pairwise error probability (PEP). We prove that the proposed DSFC can achieve full diversity of order LN, where L is the number of paths of the channel and N is the number of relay nodes, for any N and for the cases of $L = 1$ (flat, frequency-nonselective fading channel) and $L = 2$ (two-ray fading channel).**

I. Introduction

The advent of future wireless multimedia services, requiring high signal quality and high data rate, has increased the attention toward the study of wireless channels. The wireless resources, such as the bandwidth, are scarce and it is difficult to meet the high data rate requirement unless some efficient techniques are employed. Spatial diversity has proved to be an eminent candidate for achieving the signal quality and high data rate promised by the future multimedia services. Spatial diversity is also of special interest as it does not increase the overhead in the system in terms of the bandwidth and delay. The seminal work [1] revealed the increased capacity of the wireless channels by employing Multiple-Input Multiple-Output (MIMO) channels.

In wireless applications, it is affordable to have multiple antennas at the base station but it is difficult to equip the small mobile units with more than one antenna due to space constraints of the mobile units. Hence, the use of multiple antennas at the mobile units is limited. This gave rise to what is known as *cooperative diversity* in which the nodes emulate a virtual multiple element transmit antenna.

The techniques of cooperative diversity have been introduced in [2], where different protocols were proposed to achieve spatial diversity through node cooperation. Among those protocols are the decode-and-forward and amplify-and-forward protocols. The amplify-and-forward protocol does not suffer from the error propagation problem because the relays do not perform any hard-decision operation on the received signal but noise accumulates with the desired signal along the transmission path.

The main problem with the multi-node decode-and-forward protocol and the multi-node amplify-and-forward protocol is the loss in the data rate as the number of relay nodes increases. The use of orthogonal subchannels for the relay node transmissions, either through TDMA or FDMA, results in a high loss of the system spectral efficiency. This leads to the use of what is known as distributed space-time coding, where relay nodes are allowed to simultaneously transmit over the same channel by emulating a space-time code. Several works have considered the application of the existing space-time codes in a distributed fashion for the wireless relay network [3], [4], [5].

For the case of multi-path fading channels, the design of distributed space-frequency codes (DSFCs) is crucial to exploit the multi-path (frequency) diversity of the channel. The presence of multi-paths provides another mean for achieving diversity across the frequency axis. In this paper, we propose a design for distributed space-frequency codes (DSFCs) over relay channels that can exploit the multi-path diversity of wireless channels. We prove that the proposed design of DSFC can achieve full diversity of order LN, where L is the number of multi-paths per channel and N is the number of relay nodes. We prove the previous result for any number of relays N and for the cases of $L = 1$ (flat fading channel) and $L = 2$ (two-ray fading channel).

II. System Model

In this section, the system model for the distributed space-frequency coding is presented. We consider a two-hop relay channel model where there is no direct link between the source and the destination nodes. A simplified system model is depicted in Fig. 1. The system is based on orthogonal frequency division multiplexing (OFDM) modulation with K subcarriers. The channel between the source and the n-th relay node is modeled as multi-path fading channel with L paths as

$$h_{s,r_n}(\tau) = \sum_{l=1}^{L} \alpha_{s,r_n}(l)\delta(\tau - \tau_l), \tag{1}$$

where τ_l is the delay of the l-th path, and $\alpha_{s,r_n}(l)$ is the complex amplitude of the l-th path. The $\alpha_{s,r_n}(l)$'s are modeled as zero-mean complex Gaussian random variables with variance $E\left[|\alpha_{s,r_n}(l)|^2\right] = \sigma^2(l)$, where we assume symmetry

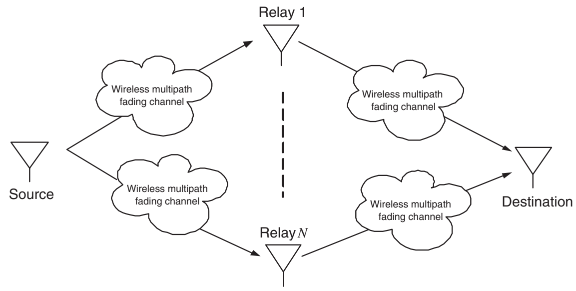

Fig. 1. Simplified system model for the distributed space-frequency codes.

between the relay nodes to simplify the analysis. The channels are normalized such that $\sum_{l=1}^{L}\sigma^2(l) = 1$. A cyclic prefix is introduced to convert the multi-path frequency-selective fading channels to flat fading sub-channels on the subcarriers.

The system has two phases as follows. In phase 1, if N relays are assigned for helping the source, the source broadcasts the information to the N relays. The received signal in the frequency domain on the k-th subcarrier at the n-th relay node is given by

$$y_{s,r_n}(k) = \sqrt{P_s}H_{s,r_n}(k)s(k) + \eta_{s,r_n}(k), \quad k = 1, \cdots, K, \tag{2}$$

where P_s is the transmitted source power, $H_{s,r_n}(k)$ is the attenuation of the source to the n-th relay channel on the k-th subcarrier, $s(k)$ is the transmitted source symbol on the k-th subcarrier, and $\eta_{s,r_n}(k)$ is the n-th relay additive white Gaussian noise on the k-th subcarrier. $\eta_{s,r_n}(k)$ is modeled as circularly symmetric complex Gaussian random variable with variance $N_0/2$ per dimension. The relay noise terms on the subcarriers can be easily seen to be statistically independent assuming that the noise terms at the input of the FFT at the relay nodes are independent.

The channel attenuation in the frequency domain, $H_{s,r_n}(k)$, is given by

$$H_{s,r_n}(k) = \sum_{l=1}^{L}\alpha_{s,r_n}(l)e^{-j2\pi(k-1)\Delta f\tau_l}, \tag{3}$$

where $\Delta f = 1/T$ is the subcarrier frequency separation, and T is the OFDM symbol duration. We assume perfect channel state information at any receiving node but no channel information at transmitting nodes. We assume that all the noise terms are independent for different receiving nodes.

The proposed DSFC is described as follows. The transmitted data from the source node is parsed into sub-blocks of size NL. Let $P = \lfloor K/NL \rfloor$ denote the number of sub-blocks in the transmitted OFDM block. The transmitted $K \times 1$ SF codeword is given as

$$\begin{aligned}\mathbf{s} &= [s(1), s(2), \cdots, s(K)]^T \\ &= [\mathbf{G}_1^T, \mathbf{G}_2^T, \cdots, \mathbf{G}_P^T, \mathbf{0}_{K-PLN}^T]^T,\end{aligned} \tag{4}$$

where $\mathbf{G}_i$ is the i-th sub-block of dimension $NL \times 1$. Zeros are padded if K is not an integer multiple of NL. For each sub-block $\mathbf{G}_i$ the n-th relay forwards only the data on L subcarriers. For example, relay 1 will only forward $[\mathbf{G}_i(1), \cdots, \mathbf{G}_i(L)]$ for all i's and send zeros on the remaining set of subcarriers. In general, the n-th relay will only forward $[\mathbf{G}_i((n-1)L+1), \cdots, \mathbf{G}_i((n-1)L+L)]$ for all i's.

In phase 2, each relay node will normalize the received signal on the subcarriers that it will forward before retransmission and will send zeros on the remaining set of subcarriers. If the k-th subcarrier is to be forwarded by the n-th relay, the relay will normalize the received signal on that subcarrier by the factor $\beta(k) = \sqrt{\frac{1}{P_s|H_{s,r_n}(k)|^2+N_0}}$ [2]. The relay nodes will use OFDM modulation for transmission to the destination node. At the destination node, the received signal on the k-th subcarrier, assuming it was forwarded by the n-th relay, is given by

$$\begin{aligned}y(k) =& H_{r_n,d}(k)\sqrt{P_r}\Bigg(\sqrt{\frac{1}{P_s|H_{s,r_n}(k)|^2+N_0}}\Big(\\ & \sqrt{P_s}H_{s,r_n}(k)s(k) + \eta_{s,r_n}(k)\Big)\Bigg) + \eta_{r_n,d}(k),\end{aligned} \tag{5}$$

where P_r is the relay node power, $H_{r_n,d}(k)$ is the attenuation of the channel between the n-th relay node and the destination node on the k-th subcarrier, and $\eta_{s,r_n}(k)$ is the destination noise on the k-th subcarrier. The $\eta_{r_n,d}(k)$'s are modeled as zero-mean, circularly symmetric complex Gaussian random variables with a variance of $N_0/2$ per dimension.

III. Pairwise Error Probability (PEP) Analysis

In this section, the PEP of the DSFC with the AAF protocol is presented. Based on the PEP analysis, code design criteria are derived.

The received signal at destination on the k-th subcarrier given by (5) can be rewritten as

$$\begin{aligned}y(k) =& H_{r_n,d}(k)\sqrt{P_r}\left(\sqrt{\frac{1}{P_s|H_{s,r_n}(k)|^2+N_0}}\sqrt{P_s}H_{s,r_n}(k)s(k)\right) \\ & + z_{r_n,d}(k),\end{aligned} \tag{6}$$

where $z_{r_n,d}(k)$ accounts for the noise propagating from the relay node as well as the destination noise. $z_{r_n,d}(k)$ follows a circularly symmetric complex Gaussian random variable with a variance $\delta_z^2(k)$ of $\left(\frac{P_r|H_{r_n,d}(k)|^2}{P_s|H_{s,r_n}(k)|^2+N_0}+1\right)N_0$. The probability density function for $z_{r_n,d}(k)$ given the channel state information (CSI) is given by

$$p(z_{r_n,d}(k)/\text{CSI}) = \frac{1}{\pi\delta_z^2(k)}\exp\left(-\frac{1}{\delta_z^2(k)}|z_{r_n,d}(k)|^2\right). \tag{7}$$

The receiver applies a *Maximum Likelihood* (ML) detector to the received signal, which is given as

$$\begin{aligned}\hat{\mathbf{s}} =& \\ & \arg\min_{\mathbf{s}}\sum_{k=1}^{K}\frac{1}{\delta_z^2(k)}\left|\mathbf{y}(k) - \frac{\sqrt{P_sP_r}H_{s,r_n}(k)H_{r_n,d}(k)}{\sqrt{P_s|H_{s,r_n}(k)|^2+N_0}}\mathbf{s}(k)\right|^2,\end{aligned} \tag{8}$$

where the n index (which is the index of the relay node) is adjusted according to the k index (which is the index of the subcarrier).

Now, sufficient conditions for the code to achieve full diversity are derived. The pdf of a received vector $\mathbf{y} = [y(1), y(2), \cdots, y(K)]^T$ given that the codeword $\mathbf{s}$ was transmitted is given by

$$p(\mathbf{y}/\mathbf{s}, \mathrm{CSI}) = \left(\prod_{k=1}^{K} \frac{1}{\pi\delta_z^2(k)}\right) \exp\left(\sum_{k=1}^{K} -\frac{1}{\delta_z^2(k)} \left| y(k) - \frac{\sqrt{P_s P_r} H_{s,r_n}(k) H_{r_n,d}(k)}{\sqrt{P_s |H_{s,r_n}(k)|^2 + N_0}} \mathbf{s}(k) \right|^2 \right). \quad (9)$$

The PEP of mistaking $\mathbf{s}$ by $\tilde{\mathbf{s}}$ can be upper bounded as [6]

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \leq E\left\{\exp\left(\lambda[\ln p(\mathbf{y}/\tilde{\mathbf{s}}) - \ln p(\mathbf{y}/\mathbf{s})]\right)\right\}, \quad (10)$$

and the relation applies for any λ, which can be selected to get the tightest bound. Any two distinct codewords $\mathbf{s}$ and $\tilde{\mathbf{s}} = [\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_2, \cdots, \tilde{\mathbf{G}}_p]^T$ will have at least one index p_0 such that $\tilde{\mathbf{G}}_{p_0} \neq \mathbf{G}_{p_0}$. We will assume that $\mathbf{s}$ and $\tilde{\mathbf{s}}$ will have only one index p_0 such that $\tilde{\mathbf{G}}_{p_0} \neq \mathbf{G}_{p_0}$, which corresponds to the worst case PEP. Averaging the PEP expression in (10) over the noise distribution given in (7) we get

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \leq E\Bigg\{ \exp\Bigg(-\lambda(1-\lambda) \sum_{n=1}^{N}\sum_{l=1}^{L} \Bigg(\frac{P_s|H_{s,r_n}(J+(n-1)L+l)|^2 P_r |H_{r_n,d}(J+(n-1)L+l)|^2}{(P_s|H_{s,r_n}(J+(n-1)L+l)|^2 + P_r|H_{r_n,d}(J+(n-1)L+l)|^2 + N_0)N_0} \Bigg) \left|\mathbf{G}_{p_0}((n-1)L+l) - \tilde{\mathbf{G}}_{p_0}((n-1)L+l)\right|^2 \Bigg) \Bigg\}, \quad (11)$$

where $J = (p_0 - 1)NL$. Take $\lambda = 1/2$ to minimize the upper bound in (11), hence, we get

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \leq E\Bigg\{ \exp\Bigg(-\frac{1}{4} \sum_{n=1}^{N}\sum_{l=1}^{L} \Bigg(\frac{P_s|H_{s,r_n}(J+(n-1)L+l)|^2 P_r |H_{r_n,d}(J+(n-1)L+l)|^2}{(P_s|H_{s,r_n}(J+(n-1)L+l)|^2 + P_r|H_{r_n,d}(J+(n-1)L+l)|^2 + N_0)N_0} \Bigg) \left|\mathbf{G}_{p_0}((n-1)L+l) - \tilde{\mathbf{G}}_{p_0}((n-1)L+l)\right|^2 \Bigg) \Bigg\}, \quad (12)$$

At high SNR, the term $\frac{P_s|H_{s,r_n}(k)|^2 P_r|H_{r_n,d}(k)|^2}{(P_s|H_{s,r_n}(k)|^2 + P_r|H_{r_n,d}(k)|^2 + N_0)N_0}$ can be approximated by $\frac{P_s|H_{s,r_n}(k)|^2 P_r|H_{r_n,d}(k)|^2}{(P_s|H_{s,r_n}(k)|^2 + P_r|H_{r_n,d}(k)|^2)N_0}$ [7], which is the scaled harmonic mean of the source-relay and relay-destination SNRs on the k-th subcarrier[1]. The scaled harmonic mean of two nonnegative numbers, a_1 and a_2, can be upper and lower bounded as

$$\frac{1}{2}\min(a_1, a_2) \leq \frac{a_1 a_2}{a_1 + a_2} \leq \min(a_1, a_2). \quad (13)$$

[1]The scaling factor is 1/2 since the harmonic mean of two number, X_1 and X_2, is defined as $\frac{2X_1X_2}{X_1+X_2}$.

Using the lower bound in (13), the PEP in (12) can be further upper bounded as

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \leq E\Bigg\{ \exp\Bigg(-\frac{1}{8} \sum_{n=1}^{N}\sum_{l=1}^{L} \min\Bigg(\frac{P_s}{N_0}|H_{s,r_n}((p_0-1)NL+(n-1)L+l)|^2, \frac{P_r}{N_0}|H_{r_n,d}((p_0-1)NL+(n-1)L+l)|^2 \Bigg) \times \left|\mathbf{G}_{p_0}((n-1)L+l) - \tilde{\mathbf{G}}_{p_0}((n-1)L+l)\right|^2 \Bigg) \Bigg\}. \quad (14)$$

If $P_r = P_s$ and SNR is defined as P_s/N_0, then the PEP is now upper bounded as

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \leq E\Bigg\{ \exp\Bigg(-\frac{1}{8} \sum_{n=1}^{N}\sum_{l=1}^{L} \min\Bigg(SNR|H_{s,r_n}((p_0-1)NL+(n-1)L+l)|^2, SNR|H_{r_n,d}((p_0-1)NL+(n-1)L+l)|^2 \Bigg) \times \left|\mathbf{G}_{p_0}((n-1)L+l) - \tilde{\mathbf{G}}_{p_0}((n-1)L+l)\right|^2 \Bigg) \Bigg\}. \quad (15)$$

A. PEP Analysis for L=1

The case of L equal to 1 corresponds to a flat, frequency-nonselective fading channel. The PEP in (15) is now given by

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \leq E\Bigg\{ \exp\Bigg(-\frac{1}{8} \sum_{n=1}^{N} \min\Bigg(SNR|H_{s,r_n}((p_0-1)NL+(n-1)L+1)|^2, SNR|H_{r_n,d}((p_0-1)NL+(n-1)L+1)|^2 \Bigg) \times \left|\mathbf{G}_{p_0}((n-1)L+1) - \tilde{\mathbf{G}}_{p_0}((n-1)L+1)\right|^2 \Bigg) \Bigg\}. \quad (16)$$

We can easily show that the random variables $SNR|H_{s,r_n}(k)|^2$ and $SNR|H_{r_n,d}(k)|^2$ follow an exponential distribution with rate $1/SNR$ for all k. The minimum of two exponential random variables is an exponential random variable with rate that is the sum of the two random variables rates. Hence, $\min\left(SNR|H_{s,r_n}(k)|^2, SNR|H_{r_n,d}(k)|^2\right)$ follows an exponential distribution with rate $2/SNR$. The

PEP upper bound is now given by

$$PEP(\mathbf{s}\rightarrow\tilde{\mathbf{s}}) \leq \prod_{n=1}^{N} \frac{1}{1+\frac{1}{16}SNR\left|\mathbf{G}_{p_0}((n-1)L+1)-\tilde{\mathbf{G}}_{p_0}((n-1)L+1)\right|^2}. \quad (17)$$

At high SNR, we neglect the 1 term in the denominator of (17). Hence, the PEP can now be upper bounded as

$$PEP(\mathbf{s}\rightarrow\tilde{\mathbf{s}}) \lesssim \left(\frac{1}{16}SNR\right)^{-N} \times \left(\prod_{n=1}^{N}\left|\mathbf{G}_{p_0}((n-1)L+1)-\tilde{\mathbf{G}}_{p_0}((n-1)L+1)\right|^2\right)^{-1}. \quad (18)$$

The result in (18) is under the assumption that the product $\prod_{n=1}^{N}\left|\mathbf{G}_{p_0}((n-1)L+1)-\tilde{\mathbf{G}}_{p_0}((n-1)L+1)\right|^2$ is non-zero. Clearly, if that product is non-zero, then the system will achieve a diversity of order NL, where L is equal to 1 in this case. From the expression in (18) the coding gain of the space-frequency code is maximized when the product $\min_{\mathbf{s}\neq\tilde{\mathbf{s}}}\prod_{n=1}^{N}\left|\mathbf{G}_{p_0}((n-1)L+1)-\tilde{\mathbf{G}}_{p_0}((n-1)L+1)\right|^2$ is maximized. This product is known as the minimum product distance [8].

B. PEP Analysis for L=2

The PEP in (15) can now be given as

$$PEP(\mathbf{s}\rightarrow\tilde{\mathbf{s}}) \leq E\Bigg\{\exp\Bigg(-\frac{1}{8}\sum_{n=1}^{N}\sum_{l=1}^{2} \min\Big(SNR|H_{s,r_n}((p_0-1)NL+(n-1)L+l)|^2, SNR|H_{r_n,d}((p_0-1)NL+(n-1)L+l)|^2\Big) \times\left|\mathbf{G}_{p_0}((n-1)L+l)-\tilde{\mathbf{G}}_{p_0}((n-1)L+l)\right|^2\Bigg)\Bigg\}, \quad (19)$$

where $L=2$. The analysis in this case is more involved since the random variables appearing in (19) are correlated. Signals transmitted from the same relay node on different subcarriers will experience correlated channel attenuations. As a first step in deriving the code design criterion, we prove that the channel attenuations, $|H_{s,r_n}(k_1)|^2$ and $|H_{s,r_n}(k_2)|^2$ for any $k_1\neq k_2$, have a bivariate Gamma distribution as their joint pdf [9]. The same applies for $|H_{r_n,d}(k_1)|^2$ and $|H_{r_n,d}(k_2)|^2$ for any $k_1\neq k_2$. The proof of this result is given in the Appendix.

To evaluate the expectation in (19) we need the expression for the joint pdf of the two random variables $M_1=\min\left(SNR|H_{s,r_n}(k_1)|^2, SNR|H_{r_n,d}(k_1)|^2\right)$ and $M_2=\min\left(SNR|H_{s,r_n}(k_2)|^2, SNR|H_{r_n,d}(k_2)|^2\right)$ for some $k_1\neq k_2$. Although M_1 and M_2 can be easily seen to be marginally exponential random variables, they are not jointly Gamma distributed. Define the random variables $X_1=SNR|H_{s,r_n}(k_1)|^2$, $X_2=SNR|H_{s,r_n}(k_2)|^2$, $Y_1=SNR|H_{r_n,d}(k_1)|^2$, and $Y_2=SNR|H_{r_n,d}(k_2)|^2$. All of these random variables are marginally exponential with rate $1/SNR$. Under the assumptions of our channel model, the pairs (X_1,X_2) and (Y_1,Y_2) are independent. Hence, the joint pdf of (X_1,X_2,Y_1,Y_2), using the result in the Appendix, is given by

$$\begin{aligned}
&f_{X_1,X_2,Y_1,Y_2}(x_1,x_2,y_1,y_2)=f_{X_1,X_2}(x_1,x_2)f_{Y_1,Y_2}(y_1,y_2)\\
&=\frac{1}{SNR^2(1-\rho_{x_1x_2})(1-\rho_{y_1y_2})}\exp\left(-\frac{x_1+x_2}{SNR(1-\rho_{x_1x_2})}\right)\\
&I_0\left(\frac{2\sqrt{\rho_{x_1x_2}}}{SNR(1-\rho_{x_1x_2})}\sqrt{x_1x_2}\right)\exp\left(-\frac{y_1+y_2}{SNR(1-\rho_{y_1y_2})}\right)\\
&I_0\left(\frac{2\sqrt{\rho_{y_1y_2}}}{SNR(1-\rho_{y_1y_2})}\sqrt{y_1y_2}\right)U(x_1)U(x_2)U(y_1)U(y_2),
\end{aligned} \quad (20)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero and $U(\cdot)$ is the Heaviside unit step function [10]. $\rho_{x_1x_2}$ is the correlation coefficient between X_1 and X_2 and similarly, $\rho_{y_1y_2}$ is the correlation coefficient between Y_1 and Y_2.

The joint cumulative distribution function (cdf) of the pair (M_1,M_2) can be computed as

$$\begin{aligned}
F_{M_1,M_2}(m_1,m_2) &\triangleq \Pr\left[M_1\leq m_1, M_2\leq m_2\right]\\
&=\Pr\left[\min\left(X_1,Y_1\right)\leq m_1, \min\left(X_2,Y_2\right)\leq m_2\right]\\
&=2\int_{y_1=0}^{m_1}\int_{x_1=y_1}^{\infty}\int_{y_2=0}^{m_2}\int_{x_2=y_2}^{\infty} f_{X_1,X_2}(x_1,x_2)\\
&\quad f_{Y_1,Y_2}(y_1,y_2)dy_1dx_1dy_2dx_2\\
&\quad +2\int_{y_1=0}^{m_1}\int_{x_1=y_1}^{\infty}\int_{x_2=0}^{m_2}\int_{y_2=x_2}^{\infty} f_{X_1,X_2}(x_1,x_2)\\
&\quad f_{Y_1,Y_2}(y_1,y_2)dy_1dx_1dx_2dy_2,
\end{aligned} \quad (21)$$

where we have used the symmetry assumption of the source-relay and relay-destination channels. The joint pdf of (M_1,M_2) can now be given as

$$\begin{aligned}
&f_{M_1,M_2}(m_1,m_2)=\frac{\partial^2}{\partial m_1\partial m_2}F_{M_1,M_2}(m_1,m_2)\\
&=2f_{Y_1,Y_2}(m_1,m_2)\int_{x_1=m_1}^{\infty}\int_{x_2=m_2}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_1dx_2\\
&+2\int_{x_1=m_1}^{\infty}\int_{y_2=m_2}^{\infty} f_{X_1,X_2}(x_1,m_2)f_{Y_1,Y_2}(m_1,y_2)dx_1dx_2.
\end{aligned} \quad (22)$$

To get the PEP upperbound in (19) we need to calculate the expectation

$$E\Big\{ \exp\Big(-\frac{1}{8} \Big(M_1 \Big|\mathbf{G}(k_1) - \tilde{\mathbf{G}}(k_1)\Big|^2 + M_2 \Big|\mathbf{G}(k_2) - \tilde{\mathbf{G}}(k_2)\Big|^2 \Big)\Big)\Big\} = \int_{m_1=0}^{\infty}\int_{m_2=0}^{\infty} \exp\Big(-\frac{1}{8}\Big(m_1 \Big|\mathbf{G}(k_1) - \tilde{\mathbf{G}}(k_1)\Big|^2 + m_2 \Big|\mathbf{G}(k_2) - \tilde{\mathbf{G}}(k_2)\Big|^2 \Big)\Big) f_{M_1,M_2}(m_1,m_2) dm_1 dm_2. \quad (23)$$

At high enough SNR $I_0\Big(\frac{2\sqrt{\rho_{x_1x_2}}}{SNR(1-\rho_{x_1x_2})}\sqrt{x_1x_2}\Big)$ is approximately 1 [10]. Using this approximation, the PEP upper bound can be approximated at high SNR as

$$PEP(\mathbf{s} \to \tilde{\mathbf{s}}) \lesssim \Big(\prod_{m=1}^{2N} \Big|\mathbf{G}_{p_0}(m) - \tilde{\mathbf{G}}_{p_0}(m)\Big|^2 \Big)^{-1} \times \Big(\frac{1}{16}(1-\rho)SNR \Big)^{-2N}, \quad (24)$$

where $\rho = \rho_{x_1x_2} = \rho_{y_1y_2}$. Again, full diversity is achieved when the product $\prod_{m=1}^{2N} \Big|\mathbf{G}_{p_0}(m) - \tilde{\mathbf{G}}_{p_0}(m)\Big|^2$ is non-zero. The coding gain of the space-frequency code is maximized when the product $\min_{\mathbf{s}\neq\tilde{\mathbf{s}}} \prod_{m=1}^{2N} \Big|\mathbf{G}_{p_0}(m) - \tilde{\mathbf{G}}_{p_0}(m)\Big|^2$ is maximized.

The analysis becomes highly involved for any $L \geq 3$. It is very difficult to get closed form expressions in this case due to the correlation among the summed terms in (15) for which no closed-form pdf expressions, similar to (20), are known [11].

We will use a linear mapping to form the transmitted subblocks, that is $\mathbf{G} = \mathbf{V}_{NL\times NL}\mathbf{s}_{\mathbf{G}}$, where $\mathbf{s}_{\mathbf{G}}$ is the $NL \times 1$ data vector transmitted in the subblock $\mathbf{G}$. $\mathbf{s}_{\mathbf{G}}$ is from QAM or PSK constellation. It was proposed in [12] and [13] to use both Hadamard transforms and Vandermonde matrices to design the $\mathbf{V}_{NL\times NL}$ matrix. The transforms based on the vandermonde matrices proved to give larger minimum product distance than the Hadamard based transforms. Two classes of optimum transforms were proposed in [12] as follows

1) If $NL = 2^k$ $(k \geq 1)$, the optimum transform is given by $\mathbf{V}_{opt} = \frac{1}{\sqrt{NL}}\text{vander}(\theta_1, \theta_2, ..., \theta_{NL})$, where $\theta_1, \theta_2, ..., \theta_{NL}$ are $\theta_n = e^{j\frac{4n-3}{2NL}\pi}$, $n = 1, 2, ..., NL$.
2) If $NL = 3.2^k$ $(k \geq 0)$, the optimum transform is given by $\mathbf{V}_{opt} = \frac{1}{\sqrt{NL}}\text{vander}(\theta_1, \theta_2, ..., \theta_{NL})$, where $\theta_1, \theta_2, ..., \theta_{NL}$ are $\theta_n = e^{j\frac{6n-1}{3NL}\pi}$, $n = 1, 2, ..., NL$.

IV. Simulation Results

In this section, we present some simulations for the proposed distributed space-frequency code. We compare the performance of the proposed codes to the DSFC with the decode-and-forward (DAF) protocol proposed in [14]. Fig. 2 shows the case of a simple two-ray, $L = 2$, channel model with a delay of $\tau = 5\mu sec$ between the two rays. The two rays have equal powers, i.e., $\sigma^2(1) = \sigma^2(2)$. The number of subcarriers is $K = 128$ with a system bandwidth of 1 MHz. We use BPSK modulation and Vandermonde based linear transformations. Fig. 2 shows the symbol error rate (SER) of the proposed DSFCs versus the SNR defined as $SNR = \frac{P_s+P_r}{N_0}$, and we use $P_s = P_r$, i.e., equal power allocation between the source and relay nodes. We simulated three cases: all channel variances are ones, relays close to source, and relays close to destination. For the case of relays close to source, the variance of any source-relay channel is taken to be 10 and the variance of any relay-destination channel is taken to be 1. For the case of relays close to destination, the variance of any source-relay channel is taken to be 1 and the variance of any relay-destination channel is taken to be 10. Fig. 3 shows the case of a simple two-ray, $L = 2$, with a delay of $\tau = 20\mu sec$ between the two rays. The simulation setup is the same as that used in Fig. 2. From Figs. 2 and 3, it is clear that DSFCs with the DAF protocol and DSFCs with the AAF protocol achieve the same diversity order. DSFCs with the DAF protocol achieve better SER performance since they deliver a less noisy code to the destination node as compared to DSFCs with the AAF protocol, where noise propagation results from the transmissions of the relay nodes. Although DSFCs with the DAF protocol have a better SER performance, DSFCs with the AAF protocol have the advantage of requiring simple processing at the relay nodes. The DAF based DSFCs require the decoding of the received source signal while the AAF based DSFCs require only scaling of the received source signal.

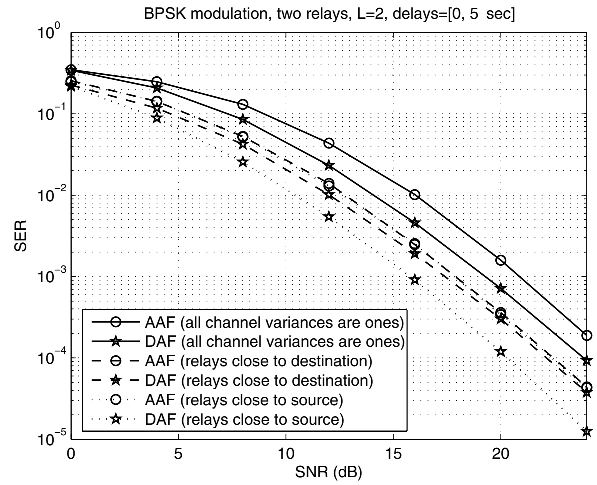

Fig. 2. SER for DSFCs for BPSK modulation, L=2, and delay=[0, 5μsec] versus SNR.

V. Conclusion

In this paper, a design of distributed space-frequency codes was proposed for the wireless relay network employing the amplify-and-forward protocol. We derive sufficient conditions for the proposed code structure, based on the PEP analysis, to achieve full diversity and maximum coding gain. We prove that the proposed codes can achieve full diversity of order LN, promised by the multi-path and cooperative diversities of the

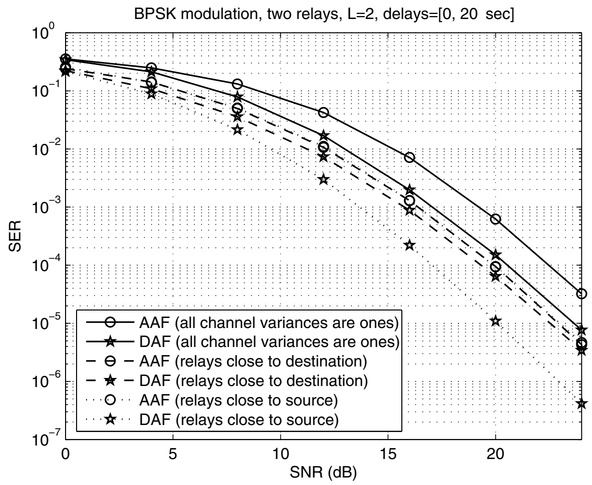

Fig. 3. SER for DSFCs for BPSK modulation, L=2, and delay=[0, 20μsec] versus SNR.

wireless relay channel, for the special cases of $L = 1$ and $L = 2$.

Appendix

Consider the two random variables $H_{s,r_n}(k_1)$ and $H_{s,r_n}(k_2)$, we will assume without loss of generality that $\tau_1 = 0$, i.e., the delay of the first path is zero. $H_{s,r_n}(k_1)$ is given by

$$\begin{aligned} H_{s,r_n}(k_1) &= \alpha_{s,r_n}(1) + \alpha_{s,r_n}(2) e^{-j2\pi(k_1-1)\Delta f \tau_2} \\ &= \Re\left(H_{s,r_n}(k_1)\right) + j\Im\left(H_{s,r_n}(k_1)\right), \end{aligned} \tag{25}$$

where $\Re(x)$, and $\Im(x)$ are the real, and imaginary parts of x, respectively. From (25) we have

$$\begin{aligned} &\Re\left(H_{s,r_n}(k_1)\right) = \Re(\alpha_{s,r_n}(1)) + \Re(\alpha_{s,r_n}(2)) \\ &\cos(2\pi(k_1-1)\Delta f\tau_2) + \Im(\alpha_{s,r_n}(2))\sin(2\pi(k_1-1)\Delta f\tau_2) \\ &\Im\left(H_{s,r_n}(k_1)\right) = \Im(\alpha_{s,r_n}(1)) + \Im(\alpha_{s,r_n}(2)) \\ &\cos(2\pi(k_1-1)\Delta f\tau_2) - \Re(\alpha_{s,r_n}(2))\sin(2\pi(k_1-1)\Delta f\tau_2). \end{aligned} \tag{26}$$

Based on the channel model presented in Section II both $\Re\left(H_{s,r_n}(k_1)\right)$ and $\Im\left(H_{s,r_n}(k_1)\right)$ are zero-mean Gaussian random variables with variance $1/2$. The correlation coefficient, ρ_{ri}, between $\Re\left(H_{s,r_n}(k_1)\right)$ and $\Im\left(H_{s,r_n}(k_1)\right)$ can be calculated as

$$\rho_{ri} = E\left\{\Re\left(H_{s,r_n}(k_1)\right)\Im\left(H_{s,r_n}(k_1)\right)\right\} = 0. \tag{27}$$

Hence, $H_{s,r_n}(k_1)$ is a circularly symmetric complex Gaussian random variable with variance $1/2$ per dimension and the same applies for $H_{s,r_n}(k_2)$. To get the joint probability distribution of $|H_{s,r_n}(k_1)|^2$ and $|H_{s,r_n}(k_2)|^2$, we can use the standard techniques of transformation of random variables. Using transformation of random variables and the fact that both $H_{s,r_n}(k_1)$ and $H_{s,r_n}(k_2)$ are circularly symmetric complex Gaussian random variables, it can be shown that $X_1 = |H_{s,r_n}(k_1)|^2$ and $X_2 = |H_{s,r_n}(k_2)|^2$ are jointly distributed according to a bivariate Gamma distribution with pdf [9], [11]

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) = &\frac{1}{1-\rho_{x_1x_2}} \exp\left(-\frac{x_1+x_2}{1-\rho_{x_1x_2}}\right) \\ &I_0\left(\frac{2\sqrt{\rho_{x_1x_2}}}{1-\rho_{x_1x_2}}\sqrt{x_1x_2}\right) U(x_1)U(x_2), \end{aligned} \tag{28}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero and $U(\cdot)$ is the Heaviside unit step function [10]. $\rho_{x_1x_2}$ is the correlation between $|H_{s,r_n}(k_1)|^2$ and $|H_{s,r_n}(k_2)|^2$ and it can be calculated as

$$\rho_{x_1,x_2} = \frac{\mathrm{Cov}\left(X_1,X_2\right)}{\sqrt{\mathrm{Var}\left(X_1\right)\mathrm{Var}\left(X_2\right)}}. \tag{29}$$

Following tedious computations, it can be shown that

$$\rho_{x_1,x_2} = \frac{1}{2} + 2\sigma^2(1)\sigma^2(2)\cos(2\pi(k_2-k_1)\Delta f\tau_2), \tag{30}$$

where the last equation applies under the assumption of having $\sigma^2(1)+\sigma^2(2) = 1$ and both, $\sigma^2(1)$ and $\sigma^2(2)$, are non zeros. From (30) it is clear that $0 \leq \rho_{x_1,x_2} \leq 1$.

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