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A structured low-rank matrix pencil for spectral estimation and system identification[1]

Javad Razavilar[a,b], Ye Li[a,b,2], K.J. Ray Liu[a,b,*]

[a] *Electrical Engineering Department and Institute for Systems Research, University of Maryland, College Park, MD 20742, USA*
[b] *Wireless Systems Research Department, AT&T Labs-Research, 791 Holmdel-Keyport Rd., Holmdel, NJ 07733-0400, USA*

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Abstract

In this paper we propose a new matrix pencil based method for estimating parameters (frequencies and damping factors) of exponentially damped sinusoids in noise. The proposed algorithm estimates the signal parameters using a matrix pencil constructed from measured data. We show that the performance of the estimation can be significantly improved by the combination of our proposed matrix pencil algorithm and the structured low rank approximation of the data matrix. Comparison of our matrix pencil method to existing matrix pencil methods as well as to polynomial methods show that our matrix pencil method is more accurate in estimating the signal parameters. It is found through computer simulations that, for exponentially damped sinusoids, our matrix pencil method is less sensitive to noise and has a lower signal-to-noise ratio (SNR) threshold. © 1998 Elsevier Science B.V. All rights reserved.

Zusammenfassung

In dieser Arbeit schlagen wir eine neue Matrixpencil-Methode für die Schätzung von Parametern (Frequenzen und Dämpfungsfaktoren) von exponentiell gedämpften Sinusfunktionen in Rauschen vor. Der vorgeschlagene Algorithmus schätzt die Signalparameter unter Verwendung eines Matrixpencils, der von gemessenen Daten konstruiert wird. Wir zeigen, daß die Schätzgüte signifikant verbessert werden kann, indem der von uns vorgeschlagene Matrixpencil-Algorithmus mit der strukturierten niederrangigen Approximation der Datenmatrix kombiniert wird. Vergleiche unserer Matrixpencil-Methode mit existierenden Matrixpencil-Methoden sowie mit Polynommethoden zeigen, daß unsere Matrixpencil-Methode bei der Schätzung der Signalparameter genauer ist. Durch Computersimulationen hat sich erwiesen, daß unsere Matrixpencil-Methode für gedämpfte Sinussignale weniger rauschempfindlich ist und eine geringere Signal-Geräuschabstandsschwelle besitzt. © 1998 Elsevier Science B.V. All rights reserved.

Résumé

Nous proposons dans cet article une nouvelle méthode basée sur le *matrix pencil* pour l'estimation des paramètres (fréquences et facteur d'amortissement) de sinusoïdes exponentiellement amorties noyées dans le bruit. L'algorithme proposé estime les paramètres du signal à l'aide d'un matrix pencil construit à partir des données. Nous montrons que les performances de l'estimation peuvent être améliorées de manière significative par la combinaison de notre algorithme de

* Corresponding author. Tel.: 1301 405 6619; fax: 1301 314 9920; e-mail: kjrliu@isr.umd.edu.
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matrix pencil et d'une approximation de rang réduit structurée de la matrice de données. La comparaison de notre méthode avec les méthodes de matrix pencil existantes ainsi qu'avec les méthodes polynomiales montrent que celle-ci est plus précise pour l'estimation des paramètres. Il est montré à l'aide de simulations sur ordinateur que, pour des sinusoïdes exponentiellement amorties, notre méthode est moins sensible au bruit et a un seuil de rapport signal sur bruit (SNR) plus bas.

Keywords: Parameter estimation; Hankel approximation; Matrix pencil

1. Introduction

High-resolution parameter estimation for exponentially damped sinusoids in the presence of additive white noise is a problem of significant interest in many signal processing applications, such as analysis of NMR[3] data, system identification, and spectral estimation. Many approaches to high-resolution spectral estimation have been proposed for stationary signals, including linear prediction (LP) techniques [3], and signal subspace methods like the MUSIC (multiple signal classification) algorithm [11]. The difficulty of parameter estimation for exponentially damped sinusoids stems from the fact that these signals are nonstationary. Therefore, most of the well-known algorithms for stationary signals cannot be applied to this kind of data directly. Polynomial methods such as KT [7] and modified KT (MKT) algorithms [7,8] are effective for estimating the frequencies and damping factors of exponentially damped sinusoids in noise. However, at low signal-to-noise (SNR) ratios, they do not provide good estimates of signal parameters. Another class of effective methods for this problem is based on the matrix pencil.

The matrix pencil method to be presented here, like the matrix pencil method proposed in [4–6], represents an alternative approach which exploits the structure of the matrix pencil of the underlying true signal (noiseless), instead of the structure of the prediction equation satisfied by the measured data as in [7,8]. In [4,6], in order to extract the true signal from the noisy sequence, singular-value decomposition (SVD) is applied to the full-rank data matrix. This would result in a rank-deficient matrix which does not have the Hankel structure of the original data matrix.

In this paper, we develop a new matrix pencil method based on the algebraic structure of the prediction matrix. In contrast to [4,6], in our algorithm we combine the matrix pencil method and the rank-deficient Hankel approximation of the data matrix, which preserves the Hankel structure of the data matrix [9]. We show that the combination of our new matrix pencil method and the structured low-rank approximation of the data matrix [1,4] outperforms the existing matrix pencil algorithms in terms of estimation accuracy and noise threshold. Computer simulations indicate that this new matrix pencil algorithm has lower noise threshold than that of the KT algorithm [7], MKT [8], and Hua–Sarkar's matrix pencil algorithms [4].

The rest of the paper is organized as follows. In Section 2, we first describe the data model and then present the new matrix pencil algorithm. Extracting the true signal from noisy data using low-rank Hankel approximation of the data matrix will be discussed in Section 3, also a summary of our matrix pencil algorithm will be presented in this section. In Section 4, some simulation results compare the performance of our algorithm with three other algorithms KT [7], MKT [8], and Hua–Sarkar's matrix pencil method [4]. Finally, in Section 5 we make some concluding remarks.

2. Development of the new matrix pencil algorithm

In this section, we present a new matrix pencil algorithm for estimating the signal parameters from a noisy exponential data sequence. Consider noisy data y_k, $k = 0, 1, \ldots, N-1$, consisting of M exponentially damped sinusoids with additive white Gaussian noise n_k, $k = 0, 1, \ldots, N-1$, which can

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[3] NMR stands for nuclear magnetic resonance, a well-known method for structural determination in molecules.

be described by

$$y_k = x_k + n_k = \sum_{m=1}^{M} c_m e^{(\alpha_m + j\omega_m)k} + n_k, \quad k = 0, \ldots, N-1, \tag{1}$$

where n_k's denote the noise samples, c_m corresponds to the amplitude of the mth sinusoid, $s_m = \alpha_m + j\omega_m$ for $m = 1, \ldots, M$, where α_m and ω_m correspond to the damping factors and frequencies of the mth sinusoid, respectively. Without loss of generality, we assume that α_m's and ω_m's are distinct and we assume that $N > 2M$.

Let $M \times M$ matrices A_l be constructed as follows:

$$A_l = \begin{bmatrix} y_l & \cdots & y_{l+M-1} \\ y_{l+1} & \cdots & y_{l+M} \\ \vdots & \vdots & \vdots \\ y_{l+M-1} & \cdots & y_{l+2M-2} \end{bmatrix} \tag{2}$$

for $l = 0, 1, \ldots, N-2M+1$. Combining Eqs. (1) and (2), we can show that

$$A_l = S^{\mathrm{T}} C \Phi^l S + W_l, \tag{3}$$

where

$$W_l = \begin{bmatrix} n_l & \cdots & n_{l+M-1} \\ n_{l+1} & \cdots & n_{l+M} \\ \vdots & \vdots & \vdots \\ n_{l+M-1} & \cdots & n_{l+2M-2} \end{bmatrix}, \tag{4}$$

$$\begin{aligned} \Phi &= \mathrm{diag}(e^{s_1}, e^{s_2}, \ldots, e^{s_M}), \\ C &= \mathrm{diag}(c_1, c_2, \ldots, c_M), \\ S &= [r(s_1), r(s_2), \ldots, r(s_M)]^{\mathrm{T}}, \\ r(s_m) &= [1, e^{s_m}, \ldots, e^{(M-1)s_m}]^{\mathrm{T}}. \end{aligned} \tag{5}$$

Now, let A_l and A_{l+1}, $l = 0, 1, \ldots, N-2M$, be two consecutive matrices constructed as given in Eq. (2). It is clear that all $M \times M$ matrices A_l's are full-rank with rank M, i.e.

$$\rho\{A_l\} = M, \quad l = 0, 1, \ldots, N-2M+1, \tag{6}$$

where $\rho\{\cdot\}$ is the matrix rank operator. Now, let us consider the matrix pencil $\{A_l - \mu A_{l+1}\}$. As will be shown in the next theorem, such a matrix pencil can be used to estimate the signal parameters (e.g. frequencies and damping factors) directly.

Theorem 1. *Given A_l and A_{l+1} of the form as in Eq. (2), where $\rho\{A_l\} = \rho\{A_{l+1}\} = M$. If there is no noise (i.e. $n_k = 0$ for $k = 0, 1, \ldots, N-1$) then for the matrix pencil $\{A_l - \mu_m A_{l+1}\}$ there exist M complex numbers μ_m, $m = 1, \ldots, M$, such that*

$$\rho\{A_l - \mu_m A_{l+1}\} = M - 1, \quad m = 1, \ldots, M, \tag{7}$$

and furthermore

$$\mu_m = e^{-s_m}, \quad m = 1, \ldots, M, \tag{8}$$

where $s_m = \alpha_m + j\omega_m$ for $m = 1, \ldots, M$. Therefore, damping factors and frequencies of the exponential data can be obtained as follows:

$$\begin{aligned} \alpha_m &= -\ln(|\mu_m|), \quad m = 1, \ldots, M, \\ \omega_m &= -\angle \mu_m, \quad m = 1, \ldots, M. \end{aligned} \tag{9}$$

Proof. From Eq. (3), since we assumed there is no noise, the matrix pencil $\{A_l - \mu_m A_{l+1}\}$ can be rewritten as follows:

$$\{A_l - \mu_m A_{l+1}\} = S^{\mathrm{T}} C (\Phi^l - \mu_m \Phi^{l+1}) S. \tag{10}$$

It was mentioned before that S and C are full-rank matrices with rank M, therefore, rank reducing numbers μ_m's should reduce the rank of $(\Phi^l - \mu_m \Phi^{l+1})$ from M to $M-1$, where $\mu_m = e^{-s_m}$, $m = 1, \ldots, M$, and s_m's are signal parameters to be estimated, i.e. $s_m = \alpha_m + j\omega_m$ for $m = 1, \ldots, M$, then we have

$$\Phi^l - \mu_m \Phi^{l+1} = \begin{bmatrix} (1 - e^{s_1 - s_m}) e^{s_1 l} & \vdots & 0 \\ \cdots & (1 - e^{s_m - s_m}) e^{s_m l} = 0 & \cdots \\ 0 & \vdots & (1 - e^{s_M - s_m}) e^{s_M l} \end{bmatrix}. \tag{11}$$

Now, from the right-hand side of Eq. (11) it is clear that $\Phi^l - \mu_m \Phi^{l+1}$ is an $M \times M$ diagonal matrix with its mth row being equal to zero.

Therefore,

$$\rho\{\Phi^l - e^{-s_m}\Phi^{l+1}\} = M - 1, \quad m = 1,2,\ldots,M. \quad (12)$$

The rest of the results follows directly and this completes the proof of the theorem. □

From another point of view we can also state that, if in Eq. (3) $W_k = 0$ for $k = 0,\ldots,N-2M+1$ (no noise) then we have

$$P = A_l^{-1}A_{l+1} = S^{-1}\Phi S, \quad l = 0,\ldots,N-2M+1. \quad (13)$$

From Eq. (13), we can see that the eigenvalues of P are the diagonal elements of Φ due to the property of similarity transformation. Therefore, from Eq. (5), the eigenvalues of $A_l^{-1}A_{l+1}$ are equal to $e^{s_1}, e^{s_2}, \ldots, e^{s_M}$ for all $l = 0,\ldots,N-2M+1$. But in the presence of noise, the W_l's in Eq. (3) will not be zero anymore and $A_l^{-1}A_{l+1}$ is not simply equal to $P = S^{-1}\Phi S$. Therefore, in the presence of measurement noise, first we have to *clean* the data sequence y_k, $k = 0,1,\ldots,N-1$, from noise so that the noise effect can be minimized.

3. Low-rank Hankel approximation

As it was pointed out before, in the presence of measurement noise, we have to reduce the noise effect before we apply the matrix pencil algorithm to the exponential data. For this purpose we construct an $L \times L$ *data matrix* R with $L = \lceil N/2 \rceil$ as follows:

$$R = \begin{bmatrix} y_0 & y_1 & \cdots & y_{L-1} \\ y_1 & y_2 & \cdots & y_L \\ \vdots & \vdots & \vdots & \vdots \\ y_{L-1} & y_L & \cdots & y_{2L-2} \end{bmatrix}. \quad (14)$$

It should be pointed out that the *data matrix* R is constructed exactly in the same manner as the matrices A_l in Eq. (2), except that R is now instead $L \times L$. It is clear that the *data matrix* R has a Hankel structure. If there is no noise, R is a rank-deficient matrix with rank M (i.e. $\rho\{R\} = M$). But in the presence of measurement noise it is full-rank with rank L ($L > M$). Using SVD to approximate R with a low-rank matrix of rank M like in [4] (where M is the number of exponential signals) will result in a matrix which is not Hankel anymore. It was shown in [8,1] that in the process of approximating R with a low-rank matrix if we preserve the Hankel structure of the matrix it will result in a better approximation of the true exponential data. In the following, we present the algorithm for rank-deficient Hankel approximation of R.

Rank-deficient Hankel approximation algorithm:
Initialization: $\hat{R}^{[0]} = R$ and $r = 0$ (r is the iteration index)

1. Compute SVD($\hat{R}^{[r]}$) $= UDV^{H}$.
2. Obtain $\bar{R} = [\bar{y}_{i,j}]_{i,j=0}^{L-1} = \sum_{k=1}^{K}\sigma_k u_k v_k^{H}$.
3. Find a Hankel matrix $\hat{R}^{[r]}$ to minimize $\|\hat{R}^{[r]} - \bar{R}\|_F$, i.e.

   $$\hat{R}^{[r]} = [\hat{y}^{[r]}(i+j)]_{i,j=0}^{L-1}, \quad (15)$$

   where

   $$\hat{y}_{i+j}^{[r]} = \frac{1}{\Gamma_{ij}}\sum_{n+m=i+j,\ 0\le n,m\le L-1}\bar{y}_{n,m}, \quad (16)$$

   in which Γ_{ij} is the number of the elements in matrix $\bar{R}$ satisfying $n+m=i+j$ in Eq. (16).
4. Repeat steps 1, 2 and 3 till the rank of $\hat{R}^{[r]} = M$ (where M is the number of signals).

Now, the sequence $\hat{y}_k$, $k = 0,1,\ldots,N-1$, given by Eq. (16) is used to construct $M \times M$ matrices as given in Eq. (2),

$$\hat{A}_l = \begin{bmatrix} \hat{y}_l & \cdots & \hat{y}_{l-M-1} \\ \hat{y}_{l+1} & \cdots & \hat{y}_{l+M} \\ \vdots & \vdots & \vdots \\ \hat{y}_{l+M-1} & \cdots & \hat{y}_{l+2M-2} \end{bmatrix} \quad (17)$$

for $l = 0,1,\ldots,N-2M+1$. Then, we have

$$\hat{A}_l = S^{T}C\Phi^{l}S + \hat{W}_l = D_l + \hat{W}_l \quad (18)$$

for $l = 0,1,\ldots,N-2M+1$, with S, C and Φ are defined as in (5) and $D_l = S^{T}C\Phi^{l}S$. To estimate the signal parameters we have to construct the following matrices:

$$\hat{P}_l = \hat{A}_l^{-1}\hat{A}_{l+1} = (D_l + \hat{W}_l)^{-1}(D_{l+1} + \hat{W}_{l+1}). \quad (19)$$

The first-order approximation of $\hat{A}_l^{-1} = (D_l + \hat{W}_l)^{-1}$ is given by

$$\hat{A}_l^{-1} \approx D_l^{-1} - D_l^{-1}\hat{W}_l D_l^{-1}. \tag{20}$$

From Eq. (20) and noting that $P = D_l^{-1}D_{l+1} = S^{-1}\Phi S$, we can rewrite Eq. (19) as follows:

$$\hat{P}_l \approx P + \hat{A}_l^{-1}\hat{W}_{l+1} - D_l^{-1}\hat{W}_l P,$$
$$l = 0,1,\dots,N-2M+1. \tag{21}$$

From (21) it is clear that $\hat{P}_l$ and P do not have the same eigenvalues. It has been shown that we can compute signal parameters directly from the eigenvalues of P, but in the presence of noise we can only access $\hat{P}_l$'s rather than P. Therefore, we will get a poorer estimate of signal parameters by considering only one of $\hat{P}_l$'s. To improve the accuracy of parameter estimation, we construct a new matrix, $\tilde{P}$, from the linear combination of $\hat{P}_l$'s as follows:

$$\tilde{P} = \sum_{l=0}^{N-2M-1} \gamma_l \hat{P}_l, \tag{22}$$

where γ_l's are the weighting factors. To make the estimate $\tilde{P}$ unbiased we must ensure that

$$\sum_{l=0}^{N-2M-1} \gamma_l = 1. \tag{23}$$

From Eqs. (23) and (21), we can rewrite Eq. (22) as

$$\tilde{P} = P + \sum_{l=0}^{N-2M+1} \gamma_l \bar{W}_l, \tag{24}$$

where $\bar{W}_l = \hat{A}_l^{-1}\hat{W}_{l+1} - D_l^{-1}\hat{W}_l P$. We would like to choose weighting factors γ_l, such that $\|\tilde{P} - P\|_2$ is minimized. Now, we are interested in the solution of the following minimization problem:

$$\min \quad \|\tilde{P} - P\|_2,$$
$$\text{subject to} \quad \sum_{l=0}^{N-2M+1} \gamma_l = 1, \quad \gamma_l \geqslant 0. \tag{25}$$

For the sake of convenience let us consider the solution of this problem for the scalar random variables first. In such case, $\tilde{P}$ and P are two scalar random variables and W_l is a sequence of scalar Gaussian noise with variance σ^2. Then, the weighting factors γ_l, the solution of the minimization problem posed in Eq. (25), are *inversely proportional* to the variance of the noise sequence σ^2. This claim can be proved easily for scalar random variables; however, the minimization problem -posed in Eq. (25) involves random matrices resulted from low rank Hankel and also other approximations. The closed-form solution for this case is uninformatively complex. Therefore, we use the fact from the scalar case that the weighting factors γ_l's should be inversely proportional to the variance of the elements of the noise matrix $\bar{W}_l$. The variance of the noise elements $\bar{w}_n$ (after the Hankel approximation process) is proportional to $1/L + 1 - |L - n|$ for $n = 0, 1, \dots, N-1$. After normalizing the weighting factors, in our simulations we have chosen γ_l's as

$$\gamma_l = \frac{|\det(\hat{A}_l)|^{2/M}(L + 1 - |L - l|)}{\sum_{j=0}^{N-2M+1}|\det(\hat{A}_j)|^{2/M}(L + 1 - |L - j|)}. \tag{26}$$

From Eqs. (26) and (22) we can compute $\tilde{P}$ and the signal parameters (frequencies and damping factors) can be estimated from the eigenvalues of $\tilde{P}$. It should be pointed out that in our algorithm the order of the model has to be determined in advance. There are various effective methods for this purpose suggested in [2,3,10,12]. A summary of our new matrix pencil algorithm for estimating the parameters of exponentially damped sinusoids is given in Table 1.

Table 1
New matrix pencil algorithm

Step 1	From the given data sequence y_k construct the data matrix R given by Eq. (14).
Step 2	Apply the rank-deficient Hankel approximation algorithm described in Section 3 to R.
Step 3	Use the sequence $\hat{y}_k$ given by (16) to construct $M \times M$ matrices $\hat{A}_l$ given by Eq. (17).
Step 4	From matrices $\hat{A}_l$ (Step 3) construct matrices $\hat{P}_l$ given by (19).
Step 5	From Eqs. (22) and (26), construct $\tilde{P}$.
Step 6	Estimate signal parameters from the eigenvalues of the matrix $\tilde{P}$ constructed in Step 5.

Remarks

- The convergence of the above iteration for Hankel approximation can be proved using the theory of composite mapping [1]. In [1], it has been shown that the exponential data satisfy the hypotheses of composite mapping theorem, and therefore, the composite mapping algorithm can be used to reduce the noise effect from the measured exponential data. The same results hold for

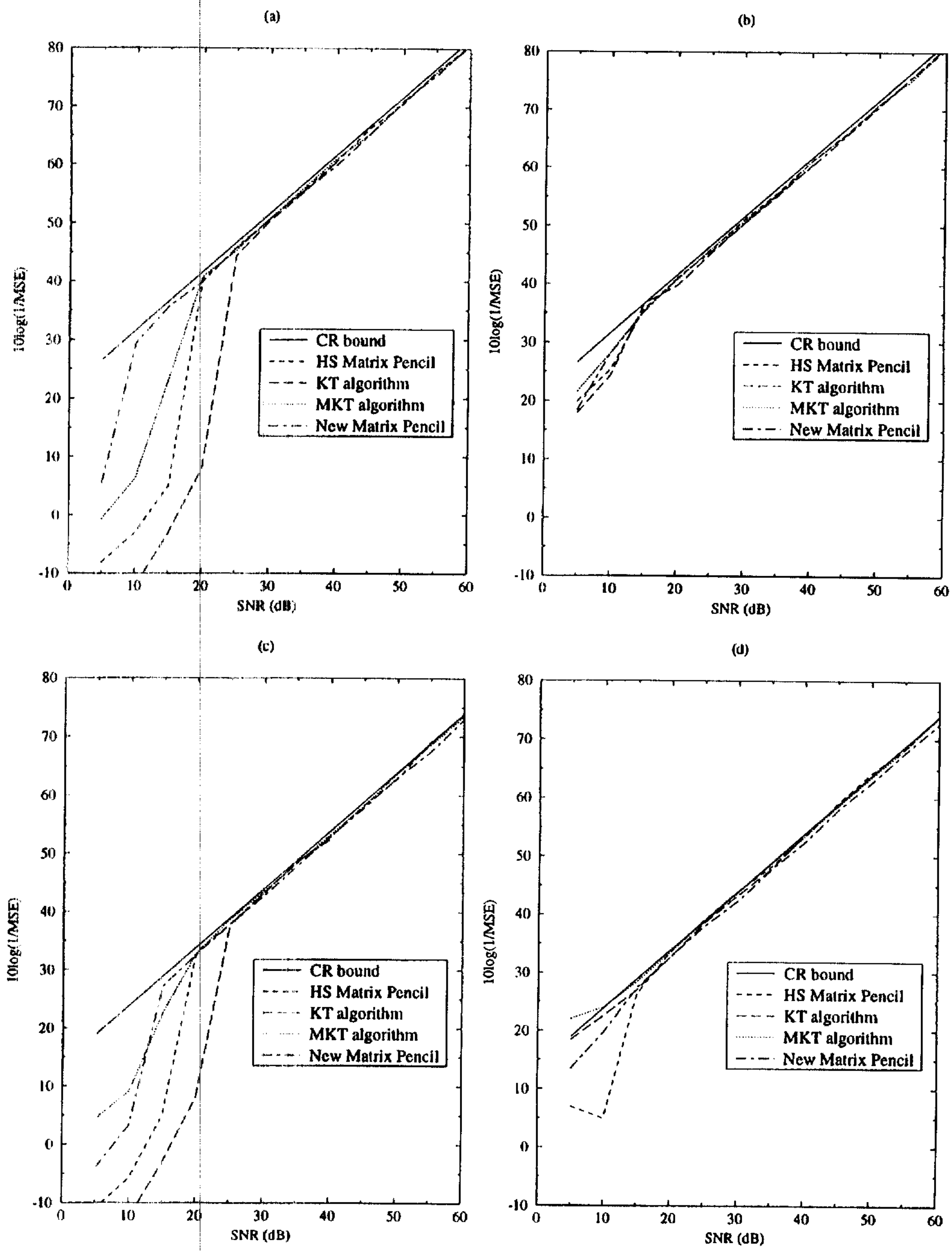

Fig. 1. The MSE of (a) ω_1, (b) α_1, (c) ω_2, (d) α_2 for KT, MKT, Hua–Sarkar's matrix pencil and the new matrix pencil algorithms when $s_1 = -0.1 + j2\pi 0.52$, $s_2 = -0.2 + j2\pi 0.42$ and $N = 25$.

damped sinusoids because they form a subset of exponential signals.

- The error generated in the low-rank Hankel approximation process is a decreasing and bounded function. Our observations confirm that for most of the cases, it practically reaches its final value in 3–5 iterations.
- It is worth mentioning that our matrix pencil algorithm has the same order of complexity as the KT algorithm [7] and Hua–Sarkar's matrix pencil algorithm [4]. As a matter of fact extra computations in our matrix pencil algorithm come from the Hankel approximation part (Step 2 in Table 1, which requires several SVDs until the algorithm converges. But for most practical cases of interest the Hankel approximation typically converges within a few iterations. For the results we have reported in this paper we performed only three iterations in the Hankel approximation algorithm and we still got very good results. Therefore, the order of the complexity in our algorithm remains the same as that of the KT algorithm [7] and Hua–Sarkar's matrix pencil method [4].

4. Computer simulation results

In this section, we will demonstrate the performance of the new matrix pencil algorithm using two examples drawn from [7].

Example 1. The purpose of the first example is to demonstrate the performance of the new matrix pencil algorithm for spectral estimation. The simulated data are generated as follows:

$$y_k = e^{s_1 k} + e^{s_2 k} + n_k \quad \text{for } k = 0, 1, \ldots, 24, \tag{27}$$

where the number of signals $M = 2$, $s_i = -\alpha_i + j2\pi f_i$ with $\alpha_1 = 0.1, f_1 = 0.52$, $\alpha_2 = 0.2$, $f_2 = 0.42$, n_k is complex white Gaussian noise with zero mean and variance σ^2. The signal-to-noise

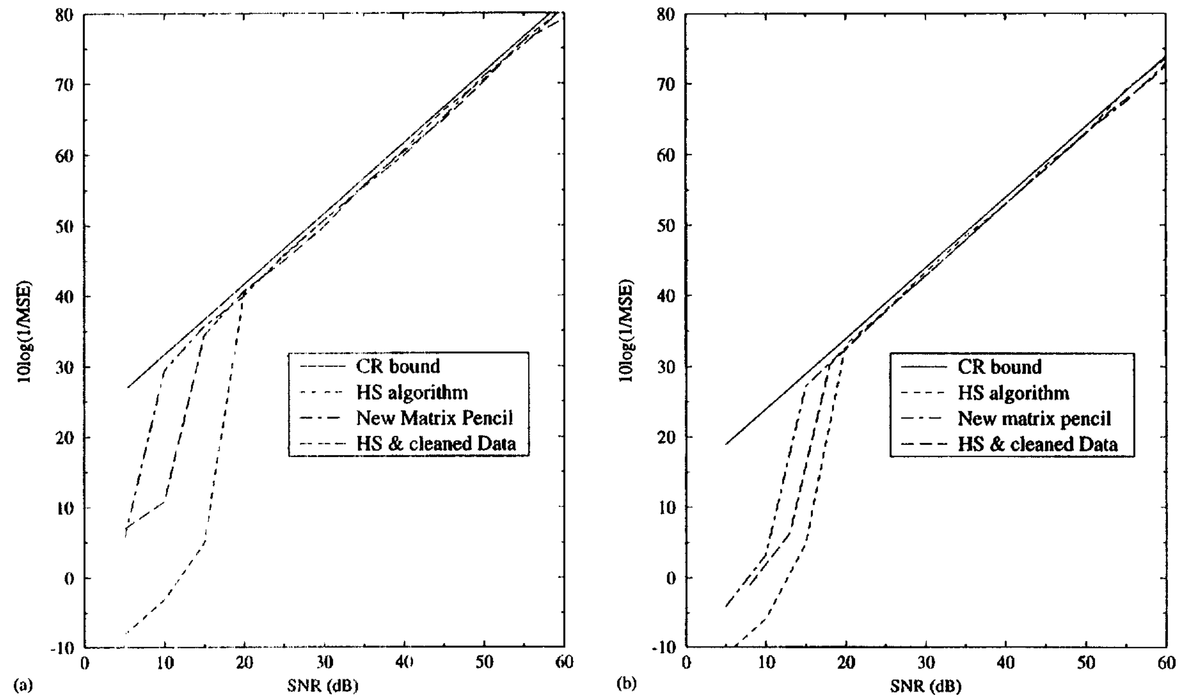

Fig. 2. The MSE of (a) ω_1, (b) ω_2 for Hua–Sarkar's matrix pencil with and without preprocessing the data and the new matrix pencil algorithms when $s_1 = -0.1 + j2\pi 0.52$, $s_2 = -0.2 + j2\pi 0.42$ and $N = 25$.

ratio (SNR) is defined as

$$\mathrm{SNR} = 10\log\left(\frac{1}{2\sigma^2}\right). \quad (28)$$

The MSEs of ω_1, α_1, ω_2 and α_2 for the KT algorithm [7], MKT [8], Hua–Sarkar's matrix pencil method [6,4], and the new matrix pencil algorithm, using the average of 500 trails, are shown in Fig. 1. Fig. 1(a) shows that the noise threshold associated with a smaller damping factor is lower than that associated with a larger damping factor as given in Fig. 1(c). Fig. 1(a) shows that when estimating ω_1 the noise threshold of the new matrix pencil algorithm is about 12 dB lower than that of the KT algorithm and is about 8 dB lower than those of MKT and Hua–Sarkar's matrix pencil algorithms. Also Fig. 1(c) shows that when estimating ω_2 the noise threshold of the new matrix pencil algorithm is about 10 dB lower than that of the KT algorithm and is about 5 dB lower than those of MKT and Hua–Sarkar's matrix pencil algorithms.

Table 2
The true and estimated poles of the transfer function $H(z)$

True poles	Method		Estimated poles SNR = 30 dB	SNR = 20 dB
$-0.2913 \pm j0.8968$	KT	Mean	$-0.2880 \pm j0.9080$	$-0.3448 \pm j0.8720$
	alg.	Std.	1.498×10^{-4}	8.583×10^{-2}
	MKT	Mean	$-0.2915 \pm j0.8970$	$-0.2917 \pm j0.8976$
	alg.	Std.	3.762×10^{-5}	3.376×10^{-4}
	New	Mean	$-0.2914 \pm j0.8964$	$-0.2914 \pm j0.8959$
	alg.	Std.	1.048×10^{-5}	1.088×10^{-4}
$0.1014 \pm j0.9579$	KT	Mean	$0.0987 \pm j0.9510$	$0.0657 \pm j0.9392$
	alg.	Std.	1.713×10^{-5}	1.406×10^{-2}
	MKT	Mean	$0.1015 \pm j0.9577$	$0.1015 \pm j0.9571$
	alg.	Std.	4.311×10^{-6}	4.556×10^{-5}
	New	Mean	$0.1014 \pm j0.9580$	$0.1014 \pm j0.9578$
	alg.	Std.	5.800×10^{-7}	6.292×10^{-6}
$0.2959 \pm j0.9292$	KT	Mean	$0.2979 \pm j0.9232$	$0.2845 \pm j0.9183$
	alg	Std.	2.784×10^{-5}	3.858×10^{-3}
	MKT	Mean	$0.2960 \pm j0.9291$	$0.2959 \pm j0.9287$
	alg	Std.	1.992×10^{-6}	1.607×10^{-5}
	New	Mean	$0.2959 \pm j0.9292$	$0.2960 \pm j0.9292$
	alg	Std.	2.295×10^{-7}	2.211×10^{-6}
$0.5630 \pm j0.8019$	KT	Mean	$0.5620 \pm j0.8175$	$0.5339 \pm j0.8442$
	alg	Std.	1.827×10^{-4}	8.359×10^{-3}
	MKT	Mean	$0.5629 \pm j0.8023$	$0.5626 \pm j0.8045$
	alg	Std.	1.059×10^{-5}	9.462×10^{-5}
	New	Mean	$0.5629 \pm j0.8018$	$0.5629 \pm j0.8020$
	alg	Std.	2.224×10^{-6}	2.233×10^{-5}
$0.9815 \pm j0.1117$	KT	Mean	$0.9853 \pm j0.1089$	$0.9798 \pm j0.1219$
	alg.	Std.	2.120×10^{-5}	1.763×10^{-2}
	MKT	Mean	$0.9816 \pm j0.1117$	$0.9821 \pm j0.1114$
	alg.	Std.	1.894×10^{-6}	1.811×10^{-5}
	New	Mean	$0.9815 \pm j0.1118$	$0.9815 \pm j0.1117$
	alg.	Std.	5.858×10^{-7}	4.988×10^{-6}

It was mentioned before that to reduce the noise effect, in our algorithm we have preprocessed the noisy data to minimize the noise effect (Step 2 in Table 1 by performing rank-deficient Hankel approximation. In order to see the effect of this preprocessing on the performance of the original Hua–Sarkar matrix pencil algorithm, we also employed the preprocessed data in the Hua–Sarkar matrix pencil method. Fig. 2 shows that this preprocessing indeed improves the performance of the H–S matrix pencil algorithm also. Nevertheless, even in this case our new matrix pencil algorithm outperforms it with 2–5 dB lower noise threshold.

Example 2. We want to estimate the poles and zeros of a linear system from noisy samples of the impulse response. The transfer function of the linear system is

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 + \sum_{k=1}^{2} b_k z^{-k}}{1 + \sum_{k=1}^{10} a_k z^{-k}}, \tag{29}$$

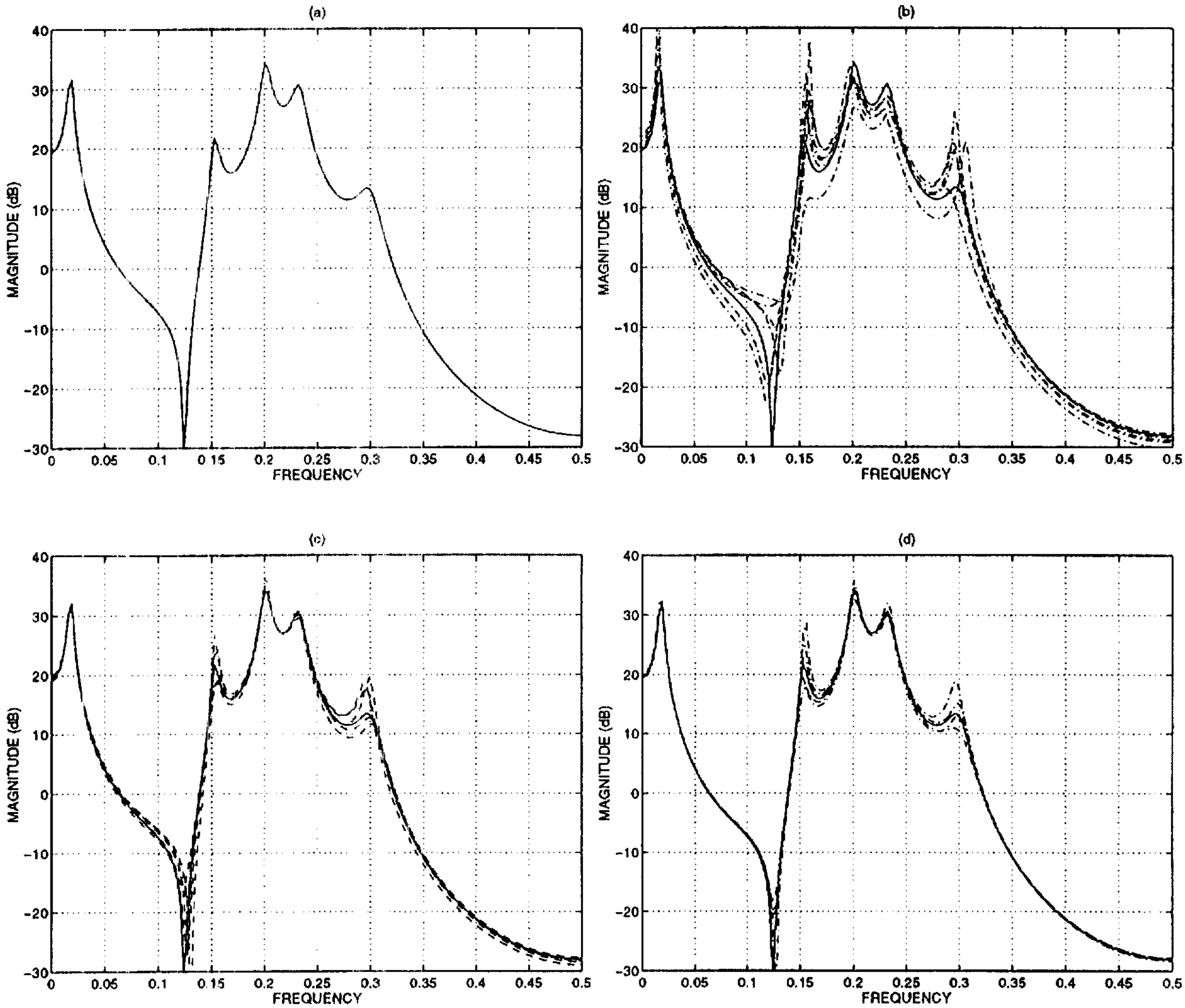

Fig. 3. (a) Magnitude of $H(z)$. (b) estimated magnitude of $\hat{H}(z)$ using KT algorithm. (c) estimated magnitude of $\hat{H}(z)$ using MKT algorithm. (d) estimated magnitude of $\hat{H}(z)$ using the new algorithm. SNR = 30 dB.

where the poles of the transfer function are shown in Table 2. The magnitude of $H(e^{j\omega})$ is shown in Fig. 3(a), which has two nulls at $\omega = \pm\pi/4$, respectively. The first forty real-valued samples of noise-corrupted impulse response are observed. The noise is a real white Gaussian process with zero mean and variance σ^2 determined by a SNR defined by

$$\mathrm{SNR} = 10\log\left(\frac{\sum_{n=0}^{39}|h(n)|^2}{N\sigma^2}\right). \tag{30}$$

In this example, the KT algorithm, the MKT algorithm, and the new matrix pencil algorithm are employed to estimate the poles of the system to get $\hat{A}(z)$. Once $\hat{A}(z)$ is obtained, $\hat{B}(z)$ can be estimated using Shanks' method [7], which first generates a sequence f_n by

$$f(n) = \mathscr{Z}^{-1}\left\{\frac{1}{\hat{A}(z)}\right\} \tag{31}$$

and then estimates b_k for $k = 0, 1, 2$ by minimizing the error

$$E = \sum_{n=0}^{39}\left|h(n) - \sum_{k=0}^{2}\hat{b}_k f(n-k)\right|^2. \tag{32}$$

The magnitude of the transfer function was estimated in ten independent trials using KT, MKT and the new matrix pencil algorithms, respectively. Fig. 3(b–d) shows the results. Table 2 shows the mean and variance of the estimated poles of the transfer function using KT, MKT and the new matrix pencil algorithms. From Fig. 3 and Table 2, it is clear that the new matrix pencil algorithm outperforms the KT and MKT algorithms.

5. Conclusions

In this paper a new matrix pencil algorithm for estimating the parameters (frequencies and damping factors) of exponentially damped sinusoids in noise is proposed. The proposed algorithm estimates the signal parameters by constructing a matrix pencil from the measured exponential data. We have shown that a better estimate can be obtained by combining the proposed matrix pencil and the structured low-rank Hankel approximation of the data matrix. Comparisons of our matrix pencil method with the polynomial methods and other matrix pencil methods show that our proposed matrix pencil method is more accurate in estimating the signal parameters.

Computer simulations indicate that, for exponentially damped sinusoids, our matrix pencil method is less sensitive to noise than the polynomial methods and existing matrix pencil methods. Simulations also confirmed that our new matrix pencil algorithm has a lower signal-to-noise ratio (SNR) threshold than those algorithms.

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