SPECTRAL ESTIMATION BASED ON STRUCTURED LOW RANK MATRIX PENCIL

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ABSTRACT

This paper proposes a new parameter estimation algorithm for damped sinusoidal signals. Parameter estimation for damped sinusoidal signals with additive white noise is a problem of significant interest in many signal processing applications, like analysis of NMR data and system identification. The new algorithm estimates the signal parameters using a matrix pencil constructed from the measured data. To reduce the noise effect, rank deficient Hankel approximation of prediction matrix is used. The performance of the new algorithm is significantly improved by structured low rank approximation of prediction matrix. Computer simulations show that the noise threshold of the new algorithm is significantly better than the existing algorithms.

1. INTRODUCTION

High resolution parameter estimation for damped sinusoidal signals in the presence of additive white noise is a problem of significant interest in many signal processing applications, like spectral analysis, analysis of NMR data and system identification. Many approaches to high resolution parameter estimation have been proposed for pure sinusoidal signals, including linear prediction (LP) techniques [1], signal subspace methods introduced by Pisarenko, and generalized by Schmidt [2] in his MUSIC (multiple signal classification) algorithm and ESPRIT [3] which provides high resolution parameter estimation by means of subspace rotational invariance techniques with com-

plexity much less than MUSIC algorithm. The difficulty of parameter estimation for damped sinusoidal signals stems from the nonstationarity of this kind of signals. Kumaresan-Tufts (KT) algorithm [4] and Hua-Sarkar's matrix pencil method [7] are well known to provide better estimations for parameters of damped sinusoidal signals. These two methods can attain Cramer-Rao bound for a certain noise threshold if the damping factors are small.

A new algorithm is proposed in this paper which estimates the signal parameters by using a matrix pencil constructed from noise corrupted data. To reduce the noise effect, the low rank Hankel approximation of the prediction matrix is used [5, 6] . It can be shown that the performance of parameter estimation algorithm can be improved if the structure of the prediction matrix is preserved after low rank approximation. This structured low rank approximation of prediction matrix [5, 6] has a great effect on the performance of the new algorithm. The performance of the new algorithm is compared with KT algorithm [4], modified KT algorithm (MKT) [6] and Hua-Sarkar's matrix pencil method [7] through the computer simulations. Computer simulation results show that, this new algorithm has lower noise threshold than KT algorithm [4], MKT [6] and Hua-Sarkar's matrix pencil algorithms [7].

2. ALGORITHM DEVELOPMENT

Consider a sequence $y(n)$ consisted of K damped sinusoidal signals and white noise $w(n)$ as

$$y(n) = \sum_{k=1}^{K} c_k e^{s_k n} + w(n), \quad n = 0, \cdots, N-1. \quad (1)$$

where K is the number of damped sinusoids, $N \geq 2K$, $s_k = -\alpha_k + j\omega_k$, $\omega_k \in [-\pi, \pi]$, $\alpha_k \geq 0$, which is called the damping factor and $w(n)$ represents the measurement noise. We first form $K \times K$ matrices as

$$\mathbf{A}_n = \begin{pmatrix} y(n) & \cdots & y(n+K-1) \\ y(n+1) & \cdots & y(n+K) \\ \vdots & \vdots & \vdots \\ y(n+K-1) & \cdots & y(n+2K-2) \end{pmatrix},$$

$$(2)$$

for $n = 0, 1, \cdots, N - 2K + 1$. From (1) it can be shown that

$$\mathbf{A}_n = \mathbf{S}^T \mathbf{C} \boldsymbol{\Phi}^n \mathbf{S} + \mathbf{W}_n, \qquad (3)$$

where in the above expression $\mathbf{W}_n$'s are noise matrices which have the same structure as data matrices $\mathbf{A}_n$, $\boldsymbol{\Phi}$ and $\mathbf{C}$ are diagonal matrices defined as $\boldsymbol{\Phi} = \mathrm{diag}(e^{s_1}, e^{s_2}, \cdots, e^{s_K})$, $\mathbf{C} = \mathrm{diag}(c_1, c_2, \cdots, c_K)$, $\mathbf{S}$ is the signal matrix defined as $\mathbf{S} = [\mathbf{r}(s_1), \mathbf{r}(s_2), \cdots, \mathbf{r}(s_K)]^T$, where $\mathbf{r}(s_k) = [1, e^{s_k}, \cdots, e^{(K-1)s_k}]^T$.

If there is no measurement noise:

$$\mathbf{A}_n^{-1} \mathbf{A}_{n+1} = \mathbf{S}^{-1} \boldsymbol{\Phi} \mathbf{S}. \qquad (4)$$

Therefore the eigenvalues of $\mathbf{A}_n^{-1} \mathbf{A}_{n+1}$ are $e^{s_1}, e^{s_2}, \cdots, e^{s_K}$, so signal parameters can be estimated from eigenvalues of $\mathbf{A}_n^{-1} \mathbf{A}_{n+1}$. Due to the existence of noise, we can only estimate $\mathbf{P} = \mathbf{S}^{-1} \boldsymbol{\Phi} \mathbf{S}$. To obtain an accurate estimation of $\mathbf{P}$, we construct the following statistics

$$\widehat{\mathbf{P}} = \sum_{n=0}^{N-2K} a_n \mathbf{A}_n^{-1} \mathbf{A}_{n+1}, \qquad (5)$$

where a_n's are the weighting factors. To make the estimation unbiased we must ensure $\sum_{n=0}^{N-2K} a_n = 1$. By selecting a_n properly a good estimation of $\widehat{\mathbf{P}}$ can be obtained, therefore the performance can be enhanced under the influence of noise.

3. HANKEL APPROXIMATION

In the presence of measurement noise, we have to reduce the noise effect before we can apply the above algorithm. In other words, we have to first approximate noisy sequence $y(n)$ with sequence $\widehat{y}(n)$ which is less noisy, and then apply the new

matrix pencil algorithm to the $\widehat{y}(n)$. For this purpose we first construct a prediction matrix from the measured data as

$$\mathbf{A}_y = \begin{pmatrix} y(0) & y(1) & \dots & y(L-1) \\ y(1) & y(2) & \dots & y(L) \\ \vdots & \vdots & \vdots & \vdots \\ y(L-1) & y(L) & \dots & y(2L-2) \end{pmatrix},$$

$$(6)$$

where $L = \lceil N/2 \rceil$ in order to have the best performance. Rank deficient Hankel approximation of the $\mathbf{A}_y$ given in [5, 6] is used to find the sequence $\widehat{y}(n)$ as follows:

Initialization: $\widehat{\mathbf{A}}_y^{[0]} = \mathbf{A}_y$ and $r = 0$ (r is the iteration index)

1) Compute $\quad SVD(\widehat{\mathbf{A}}_y^{[r]}) = \mathbf{U}\mathbf{D}\mathbf{V}^H$

2) Obtain $\bar{\mathbf{A}}_y = [\bar{y}_{i,j}]_{i,j=0}^{L-1} = \sum_{k=1}^{K} \sigma_k \mathbf{u}_k \mathbf{v}_k^H$.

3) Find a Hankel matrix $\widehat{\mathbf{A}}_y^{[r]}$ to minimize $\| \widehat{\mathbf{A}}_y^{[r]} - \bar{\mathbf{A}}_y \|_F$, (the best least square fit to $\bar{\mathbf{A}}_y$)

$$\widehat{\mathbf{A}}_y^{[r]} = [\widehat{y}^{[r]}(i+j)]_{i,j=0}^{L-1}, \qquad (7)$$

where

$$\widehat{y}_{i+j}^{[r]} = \frac{1}{\Gamma} \sum_{n+m=i+j,\; 0 \leq n,m \leq L-1} \bar{y}_{n,m}. \qquad (8)$$

in which Γ is the number of the elements in matrix $\bar{\mathbf{A}}_y$ satisfying $n + m = i + j$ in (8).

4) Repeat steps **1, 2** *and* **3** till the rank of $\widehat{\mathbf{A}}_y^{[r]} = K$ (where K is the number of signals).

In [5] it is formally proved that this process converges to a rank deficient Hankel matrix. The sequence $\widehat{y}(n)$ resulted from the above rank deficient Hankel approximation algorithm is used in the new algorithm instead of $y(n)$ to estimate the signal parameters effectively.

Using the above Hankel approximation method for noise reduction in our new matrix pencil algorithm we have shown that the best weighting factors a_n in (5) are as follows

$$a_n = \frac{|det(\mathbf{A}_n)|^{2/K}(L + 1 - |L - n|)}{\sum_k |det(\mathbf{A}_k)|^{2/K}(L + 1 - |L - k|)}. \qquad (9)$$

4. SIMULATION EXAMPLES

In this section, we will demonstrate the performance of the new matrix pencil algorithm by two examples drawn from [4].

Example 1:
The simulated data are generated by

$$y(n) = e^{s_1 n} + e^{s_2 n} + w(n), \quad for \ n = 0, 1, \cdots, 24. \tag{10}$$

where $s_1 = -0.1 + j2\pi(0.52)$, $s_2 = -0.2 + j2\pi(0.42)$, $w(n)$ is complex white Gaussian noise with zero-mean and variance σ^2, related to SNR by SNR $= 10\log(\frac{1}{2\sigma^2})$. The MSE's of ω_1, α_1, ω_2 and α_2 for KT [4] algorithm, MKT [6], Hua-Sarkar's matrix pencil method [7] and the new algorithm, using the average of 500 trails, are shown in Figure 1. From Figure 1 (a), the noise threshold for estimated frequency associated with smaller damping factor has lower noise threshold than the estimated frequency associated with larger damping factor, Figure 1 (c). For ω_1 with damping factor $\alpha_1 = 0.1$, from Figure 1 (a), the noise threshold of the new matrix pencil algorithm is about 12 dB lower than KT algorithm and about 8 dB is lower than MKT and Hua-Sarkar's matrix pencil algorithms. For ω_2 with damping factor $\alpha_1 = 0.2$, from Figure 1 (c), the noise threshold of the new matrix pencil algorithm is about 10 dB lower than KT algorithm and about 5 dB is lower than MKT and Hua-Sarkar's matrix pencil algorithms.

Example 2:
We are going to estimate the poles and zeros of a linear system from its noise corrupted samples of the impulse response. The transfer function of the linear system is

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 + \sum_{k=1}^{2} b_k z^{-k}}{1 + \sum_{k=1}^{10} a_k z^{-k}}, \tag{11}$$

where poles of the transfer function are shown in Table 1. The magnitude of $H(e^{j\omega})$ is shown in Figure 2 (a), which has two nulls at $\omega = \pm\pi/4$ respectively.

The first forty real valued samples of noise corrupted impulse response are observed. The noise is real white Gaussian with zero-mean and variance σ^2 determined by SNR, defined as

$$SNR = 10log(\frac{\sum_{n=0}^{39} \mid h(n) \mid^2}{N\sigma^2}), \tag{12}$$

KT algorithm, MKT algorithm and the new algorithm are applied to estimate the poles of the system, consequently, the denominator of $\hat{A}(z)$. Once $\hat{A}(z)$ is obtained, $B(z)$ can be estimated using Shanks' method [8], which first generates a sequence f_n by

$$f(n) = \mathcal{Z}^{-1}\{\frac{1}{\hat{A}(z)}\}, \tag{13}$$

and then estimate b_k for $k = 0$, 1, 2 by minimizing the error

$$E = \sum_{n=0}^{39} \mid h(n) - \sum_{k=0}^{2} \hat{b}_k f(n-k) \mid^2 . \tag{14}$$

Figure 2 (b), (c) and (d) show 10 trails of magnitudes of estimated transfer function using KT, MKT and the new matrix pencil algorithms respectively. Also Table 1 illustrates the mean and variance of the estimated poles of transfer function using KT, MKT and the new matrix pencil algorithms. From Figure 2 and Table 1, it is clear that the new matrix pencil algorithm gives more accurate estimate of transfer function $H(z)$ than KT and MKT algorithms.

5. CONCLUSION

In this paper a novel parameter estimation algorithm for exponentially damped sinusoidal signals is proposed. Signal parameters are estimated by a matrix pencil constructed from the given data. The noise effect is reduced significantly by using low rank Hankel approximation of prediction matrix. Computer simulations demonstrate that the new algorithm can estimate the parameters of exponentially damped sinusoids effectively.

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Table 1. The true and estimated poles of the transfer function $H(z)$

True Poles	Method		Estimated Poles	
			$SNR = 30dB$	$SNR = 20dB$
−0.2913 ±j0.8968	KT alg.	Mean	−0.2880 ± j0.9080	−0.3448 ± j0.8720
		Std.	1.498 × 10^{-4}	8.583 × 10^{-2}
	MKT alg.	Mean	−0.2915 ± j0.8970	−0.2917 ± j0.8976
		Std.	3.762 × 10^{-5}	3.376 × 10^{-4}
	New alg.	Mean	−0.2914 ± j0.8964	−0.2914 ± j0.8959
		Std.	1.048 × 10^{-5}	1.088 × 10^{-4}
0.1014 ±j0.9579	KT alg.	Mean	0.0987 ± j0.9510	0.0657 ± j0.9392
		Std.	1.713 × 10^{-5}	1.406 × 10^{-2}
	MKT alg.	Mean	0.1015 ± j0.9577	0.1015 ± j0.9571
		Std.	4.311 × 10^{-6}	4.556 × 10^{-5}
	New alg.	Mean	0.1014 ± j0.9580	0.1014 ± j0.9578
		Std.	5.800 × 10^{-7}	6.292 × 10^{-6}
0.2959 ±j0.9292	KT alg.	Mean	0.2979 ± j0.9232	0.2845 ± j0.9183
		Std.	2.784 × 10^{-5}	3.858 × 10^{-3}
	MKT alg.	Mean	0.2960 ± j0.9291	0.2959 ± j0.9287
		Std.	1.992 × 10^{-6}	1.607 × 10^{-5}
	New alg.	Mean	0.2959 ± j0.9292	0.2960 ± j0.9292
		Std.	2.295 × 10^{-7}	2.211 × 10^{-6}
0.5630 ±j0.8019	KT alg.	Mean	0.5620 ± j0.8175	0.5339 ± j0.8442
		Std.	1.827 × 10^{-4}	8.359 × 10^{-3}
	MKT alg.	Mean	0.5629 ± j0.8023	0.5626 ± j0.8045
		Std.	1.059 × 10^{-5}	9.462 × 10^{-5}
	New alg.	Mean	0.5629 ± j0.8018	0.5629 ± j0.8020
		Std.	2.224 × 10^{-6}	2.233 × 10^{-5}
0.9815 ±j0.1117	KT alg.	Mean	0.9853 ± j0.1089	0.9798 ± j0.1219
		Std.	2.120 × 10^{-5}	1.763 × 10^{-2}
	MKT alg.	Mean	0.9816 ± j0.1117	0.9821 ± j0.1114
		Std.	1.894 × 10^{-6}	1.811 × 10^{-5}
	New alg.	Mean	0.9815 ± j0.1118	0.9815 ± j0.1117
		Std.	5.858 × 10^{-7}	4.988 × 10^{-6}
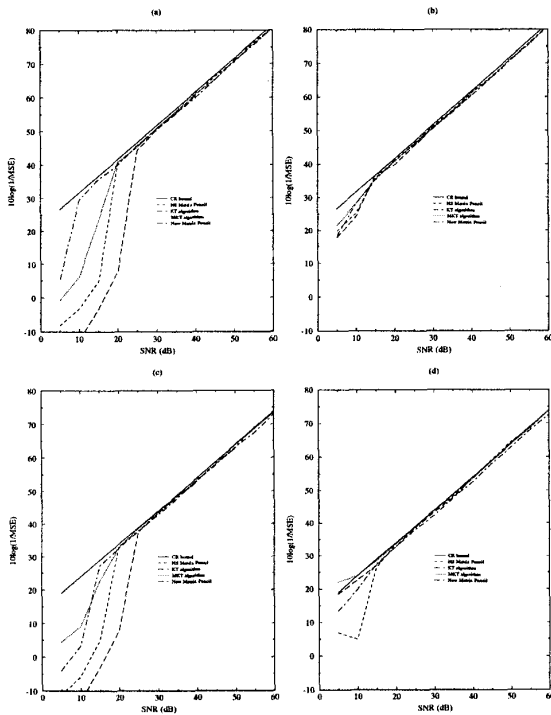
Figure 1. The MSE of (a) ω_1, (b) α_1, (c) ω_2 (d) α_2 for *KT*, MKT, Hua-Sarkar's matrix pencil and the new matrix pencil algorithms when $s_1 = -0.1 + j2\pi0.52$, $s_2 = -0.2 + j2\pi0.42$ and $N = 25$.
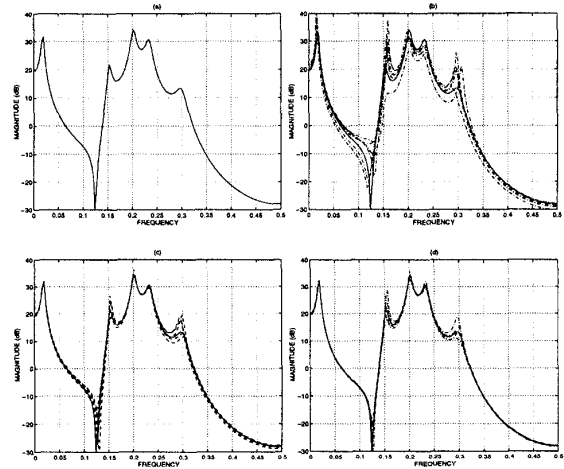

Figure 2. (a) Magnitude of $H(z)$, (b) estimated magnitude of $\widehat{H}(z)$ using KT algorithm, (c) estimated magnitude of $\widehat{H}(z)$ using MKT algorithm, (d) estimated magnitude of $\widehat{H}(z)$ using the new algorithm, SNR=30dB.