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Fast Toeplitz-Hankel Orthogonalization for High-Resolution Spectral Estimation

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Abstract

High-resolution spectral estimation is an important subject in many applications of modern signal processing. The fundamental problem in applying various high-resolution spectral estimation algorithms is the computational complexity. Recently, the truncated QR methods have been shown to be comparable to the SVD-based methods for the sinusoidal frequency estimation based on the forward-backward linear prediction (FBLP) model. However, without considering the special structure of the FBLP matrix, the QR decomposition (QRD) of the FBLP matrix has the computational complexity of $\frac{2}{3}(6m-n)n^2+O(n^2)$ for a $2m \times n$ FBLP matrix. Here we propose a fast algorithm to perform the QRD of the FBLP matrix. It is based on exploring the special Toeplitz-Hankel form of the FBLP matrix. The computational complexity is then reduced to $10n^2+4mn+O(n)$. The fast algorithm can also be easily implemented onto a linear systolic array. The number of time steps required is further reduced to $2m+n-1$ by using the parallel implementation.

1 Introduction

High-resolution spectral estimation is an important subject in many applications of modern signal processing. The fundamental problem in applying various high-resolution spectral estimation algorithms is the computational complexity. In the pioneering paper of Tufts and Kumaresan [1], a SVD-based method for solving the forward-backward linear prediction(FBLP) least-squares(LS) problem was used to resolve the frequencies of closely spaced sinusoids from a limited amount of data samples. By imposing an excessive order in the FBLP model and then truncating small singular values to zero, this truncated SVD method yields a low SNR threshold and greatly suppresses spurious frequencies. However, the massive computations required by SVD makes it unsuitable for *real time* super-resolution applications.

Recently, the truncated QR methods [4] have been shown to be comparable to the SVD-based methods in various situations. It is very effective for the sinusoidal frequency estimation based on the forward-backward linear prediction (FBLP) model. However, without considering the special structure of the FBLP matrix, the QR decomposition (QRD) of the FBLP matrix has the computational complexity of $O(n^3)$.

Seeking fast algorithms for specially structured matrices have captured lots of attention recently, especially the Toeplitz-structured matrices [2,3,6,7,8,9,10]. However, exploring the special structure of the FBLP matrix for fast algorithm implementation has not yet been considered so far. Here we propose a fast algorithm to perform the QRD of the FBLP matrix. The computational cost of the truncated QR methods can be further reduced from $O(n^3)$ to $O(n^2)$ which makes it more attractive than the SVD-based methods.

This paper is organized as follows: The basic properties and the special structure of the FBLP matrix are considered in Section 2. The fast algorithm based on the Toeplitz-Hankel structure is presented in Section 3, and finally the parallel implementation is considered in Section 4.

2 Forward-Backward Linear Prediction

2.1 Forward Linear Prediction

Suppose we observe a time sequence $u(i-1), u(i-2), \cdots, u(i-M)$, and would like to predict $u(i)$ based on a linear LS estimation. The forward linear prediction problem is to minimize the sum of the forward prediction-error energy,

$$\min_{\underline{w}_f} \sum_{i=M+1}^{N} |e_f(i)|^2, \qquad (1)$$

where

$$e_f(i) = u(i) - \underline{w}_f^T \underline{u}(i-1),$$
$$\underline{u}^T(i-1) = [u(i-1), u(i-2), \cdots, u(i-M)],$$

and $\underline{w}_f \in \Re^{M\times 1}$ is a forward prediction weight vector.

2.2 Backward Linear Prediction

On the other hand, for the backward linear prediction, we observe a sequence $u(i-M+1), u(i-M+2), \cdots, u(i)$, and would like to predict $u(i-M)$ based on a linear LS estimation. The backward linear prediction problem is to optimize the criterion,

$$\min_{\underline{w}_b} \sum_{i=M+1}^{N} |e_b(i)|^2, \tag{2}$$

where

$$e_b(i) = u(i-M) - \underline{w}_b^T \underline{u}^B(i),$$

$$\underline{u}^{BT}(i) = [u(i-M+1), u(i-M+2), \cdots, u(i)],$$

and $\underline{w}_b \in \Re^{M\times 1}$ is a backward prediction weight vector. Here B denotes the backward arrangement of a vector, that is, $\underline{u}^{BT}(i)$ is a backward arrangement of the vector $\underline{u}(i)$.

2.3 Forward-Backward Linear Prediction

To obtain a smoother result, we can combine both the forward and backward linear prediction together. This we call the forward-backward linear prediction (FBLP) method which is to minimize the sum of the FBLP errors energy,

$$\min_{\underline{w}} \sum_{i=M+1}^{N} [|e_f(i)|^2 + |e_b(i)|^2]. \tag{3}$$

The data matrix $A \in \Re^{2(N-M)\times M}$ of the FBLP method is given by

$$A = \begin{bmatrix} u(1) & u(2) & \cdots & u(M) \\ u(2) & u(3) & \cdots & u(M+1) \\ \vdots & \vdots & & \vdots \\ u(N-M) & u(N-M+1) & \cdots & u(N-1) \\ ---- & ---- & ---- & ---- \\ u(M+1) & u(M) & \cdots & u(2) \\ \vdots & \vdots & & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-M) \\ u(N) & u(N-1) & \cdots & u(N-M+1) \end{bmatrix} \tag{4}$$

and the desired response vector $\underline{b}$ is

$$\underline{b}^T = [u(M+1), \cdots, u(N), \vdots, u(1), \cdots, u(N-M)]. \tag{5}$$

The FBLP method is to solve the following LS problem,

$$A\underline{w} \cong \underline{b}. \tag{6}$$

An augmented form of the FBLP method can be obtained by putting A and $\underline{b}$ together as

$$[A \vdots \underline{b}] = \begin{bmatrix} u(1) & u(2) & \cdots & u(M) & u(M+1) \\ u(2) & u(3) & \cdots & u(M+1) & u(M+2) \\ \vdots & \vdots & & \vdots & \vdots \\ u(N-M) & u(N-M+1) & \cdots & u(N-1) & u(N) \\ ---- & ---- & ---- & ---- & ---- \\ u(M+1) & u(M) & \cdots & u(2) & u(1) \\ \vdots & \vdots & & \vdots & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-M) & u(N-M-1) \\ u(N) & u(N-1) & \cdots & u(N-M+1) & u(N-M) \end{bmatrix} = \begin{bmatrix} H \\ --- \\ T \end{bmatrix}. \tag{7}$$

It is interesting to see that there is a special structure in this augmented matrix. The matrix can be partitioned into two parts; the upper submatrix is of Hankel structure and the lower one is of Toeplitz structure. Furthermore, both matrices can be related as follows,

$$H = TJ, \tag{8}$$

where $J \in \Re^{(M+1)\times(M+1)}$ is an exchange matrix given by

$$J = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

The matrix of the form as given in (7) is called the *Toeplitz-Hankel* matrix. As we can see, the augmented matrix of the FBLP problem is of the Toeplitz-Hankel form with a special property, *i.e.*

$$K = [A \vdots \underline{b}] = \begin{bmatrix} H \\ ---- \\ T \end{bmatrix} = \begin{bmatrix} TJ \\ ---- \\ T \end{bmatrix}. \tag{9}$$

This special property can be used for developing a fast algorithm that will be considered in the next Section.

3 Fast Orthogonalization for Toeplitz-Hankel Matrix

By using the truncated QR method for the high-resolution AR spectral estimation, the key computational issue is to solve the FBLP LS problem based on the QR decomposition (QRD). Without considering the special structure, a conventional QRD requires $\approx 4(N-M)M^2 + O(M^2)$ multiplications to obtain the upper triangular matrix R. This is on the order of $O(M^3)$ since usually $N \gg M$. Thus, a reasonable approach is to find a fast algorithm for the FBLP LS problem by exploring its special Toeplitz-Hankel structure. This problem has not been considered so far, though the LS problem with Toeplitz structure has been studied extensively [2,3,6,7,8,9,10]. Basically, our approach shares the same spirit as that of Bojanczyk, Brent, and de Hoogs' work on the QRD of Toeplitz matrices [2].

The Toeplitz part of the Toeplitz-Hankel matrix can be partitioned as

$$T = \begin{bmatrix} u(M+1) & \underline{x}^T \\ \underline{y} & \tilde{T} \end{bmatrix} = \begin{bmatrix} \bar{T} & \underline{u} \\ \underline{v}^T & u(N-M) \end{bmatrix}, \tag{10}$$

where

$$\tilde{T} = \begin{bmatrix} u(M+1) & u(M) & \cdots & u(2) \\ u(M+2) & u(M+1) & \cdots & u(3) \\ \vdots & \vdots & & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-M) \end{bmatrix},$$

$$\underline{x}^T = [u(M), \cdots, u(2), u(1)],$$
$$\underline{y}^T = [u(M+2), \cdots, u(N-1), u(N)],$$
$$\underline{u}^T = [u(1), u(2), \cdots, u(N-M-1)],$$
$$\underline{v}^T = [u(N), u(N-1), \cdots, u(N-M+1)],$$

and the Hankel part of the Toeplitz-Hankel matrix can be partitioned as

$$H = TJ = \begin{bmatrix} \underline{u} & \tilde{H} \\ u(N-M) & \underline{v}^{BT} \end{bmatrix} = \begin{bmatrix} \underline{x}^{BT} & u(M+1) \\ \tilde{H} & \underline{y} \end{bmatrix}, \tag{11}$$

where

$$\tilde{H} = \tilde{T}J = \begin{bmatrix} u(2) & u(3) & \cdots & u(M+1) \\ u(3) & u(4) & \cdots & u(M+2) \\ \vdots & \vdots & & \vdots \\ u(N-M) & u(N-M+1) & \cdots & u(N-1) \end{bmatrix},$$

$$\underline{v}^{BT} = [u(N-M+1), \cdots, u(N-1), u(N)] = \underline{v}^T J,$$
$$\underline{x}^{BT} = [u(1), u(2), \cdots, u(M)] = \underline{x}^T J.$$

Again, here B denotes the backward arrangement of a vector.

Now, from the above partitions, the Toeplitz-Hankel matrix K can be partitioned as follows,

$$K = \begin{bmatrix} \underline{u} & \tilde{H} \\ u(N-M) & \underline{v}^{BT} \\ ---- & ---- \\ u(M+1) & \underline{x}^T \\ \underline{y} & \tilde{T} \end{bmatrix}, \tag{12}$$

and

$$K^T K = \begin{bmatrix} \underline{u}^T\underline{u} + u^2(N-M) + u^2(M+1) + \underline{y}^T\underline{y} & \underline{u}^T\tilde{H} + u(N-M)\underline{v}^{BT} + u(M+1)\underline{x}^T + \underline{y}^T\tilde{T} \\ \tilde{H}^T\underline{u} + \underline{v}^B u(N-M) + \underline{x}u(M+1) + \tilde{T}^T\underline{y} & \tilde{H}^T\tilde{H} + \tilde{T}^T\tilde{T} + \underline{v}^B\underline{v}^{BT} + \underline{x}\underline{x}^T \end{bmatrix}. \tag{13}$$

Also, the matrix K can be partitioned as

$$K = \begin{bmatrix} \underline{x}^{BT} & u(M+1) \\ \tilde{H} & \underline{y} \\ ---- & ---- \\ \tilde{T} & \underline{u} \\ \underline{v}^T & u(N-M) \end{bmatrix}, \tag{14}$$

and with this partition, we have

$$K^T K = \begin{bmatrix} \underline{x}^B\underline{x}^{BT} + \tilde{H}^T\tilde{H} + \tilde{T}^T\tilde{T} + \underline{v}\underline{v}^T & \underline{x}^B u(M+1) + \tilde{H}^T\underline{y} + \tilde{T}^T\underline{u} + \underline{v}u(N-M) \\ u(M+1)\underline{x}^{BT} + \underline{y}^T\tilde{H} + \underline{u}^T\tilde{T} + u(N-M)\underline{v}^T & u^2(M+1) + \underline{y}^T\underline{y} + \underline{u}^T\underline{u} + u^2(N-M) \end{bmatrix}. \tag{15}$$

Let the QRD of matrix K be $K = QR$, where $R \in \Re^{(M+1)\times(M+1)}$ is an upper triangular matrix and it can be partitioned as follows,

$$R = \begin{bmatrix} r_{11} & \underline{r}_1^T \\ \underline{0} & R_b \end{bmatrix} = \begin{bmatrix} R_t & \underline{r}_2 \\ \underline{0}^T & r_{M+1,M+1} \end{bmatrix}, \tag{16}$$

where $R_b \in \Re^{M\times M}$ is a principle bottom submatrix of R, $R_t \in \Re^{M\times M}$ is a principle top submatrix of R, and

$$\underline{r}_1^T = [r_{12}, r_{13}, \cdots, r_{1,M+1}],$$
$$\underline{r}_2^T = [r_{1,M+1}, r_{2,M+1}, \cdots, r_{M,M+1}].$$

Note that both R_b and R_t are upper triangular matrices. Since the matrix Q is orthogonal, we have

$$K^T K = R^T R, \tag{17}$$

and

$$R^T R = \begin{bmatrix} r_{11}^2 & r_{11}\underline{r}_1^T \\ \underline{r}_1 r_{11} & \underline{r}_1\underline{r}_1^T + R_b^T R_b \end{bmatrix} = \begin{bmatrix} R_t^T R_t & R_t^T \underline{r}_2 \\ \underline{r}_2^T R_t & \underline{r}_2^T\underline{r}_2 + r_{M+1,M+1}^2 \end{bmatrix}. \tag{18}$$

Define

$$\tilde{K} = \begin{bmatrix} \tilde{H} \\ --- \\ \tilde{T} \end{bmatrix}, \tag{19}$$

we then have

$$\tilde{K}^T\tilde{K} = \tilde{H}^T\tilde{H} + \tilde{T}^T\tilde{T}. \tag{20}$$

From the lower right submatrices of (13) and (18), we obtain

$$R_b^T R_b + \underline{r}_1\underline{r}_1^T = \tilde{K}^T\tilde{K} + \underline{x}\underline{x}^T + \underline{v}^B\underline{v}^{BT}. \tag{21}$$

Also, from the upper left submatrices of (15) and (18), we have

$$R_t^T R_t = \tilde{K}^T\tilde{K} + \underline{v}\underline{v}^T + \underline{x}^B\underline{x}^{BT}. \tag{22}$$

Substituting (22) to (21), we obtain the relation between R_b and R_t as given by

$$R_b^T R_b = R_t^T R_t + \underline{x}\underline{x}^T - \underline{x}^B\underline{x}^{BT} + \underline{v}^B\underline{v}^{BT} - \underline{v}\underline{v}^T - \underline{r}_1\underline{r}_1^T. \tag{23}$$

This equation provides a recursive generation of the upper triangular matrix R. Suppose the first k rows of R_t are available, by performing two rank-1 updatings and three rank-1 downdatings, we can obtain the kth row of R_b. This row is, in fact, identical to the $(k+1)$th row of R_t (except the last element). To start this recursion, the first row of R_t must be obtained initially. This can be done by a sequence of Givens rotations on the matrix A to zero out the first column of A except its leading element on the diagonal. The computational cost is $4(N-M)M$ multiplications (since only halh of the rotation needed to be done) for this initialization. Following this, the recursion in (23) is then started. As there are five rotation-like up/downdatings, the computational cost is $10M^2+O(M)$ (for multiplication). Therefore, the total computational complexity is $10M^2 + 4(N-M)M$ (for multiplication) for a $2(N-M) \times M$ Toeplitz-Hankel matrix. As mentioned before, without considering the special structure, by using the conventional QRD, the computational complexity is of $\approx 4M^3 + O(M^2)$. Obviously, the proposed fast algorithm has an improvement of an order of magnitude. In general, for the QRD of a $2m \times n$ Toeplitz-Hankel matrix, the fast algorithm needs $10n^2 + 4mn + O(n)$ multiplications, while a conventional implementation needs $\frac{2}{3}(6m-n)n^2 + O(n^2)$, where $m \gg n$.

4 Parallel Implementation

The fast algorithm obtained in the previous Section not only reduces the computational complexity, but is also amenable for parallel implementation. From the fact that the first row of the upper triangular matrix R has to be obtained first, a linear array can be used to rotate the matrix A such that the first column can be zeroed out and eventually the first row of the matrix R is kept in the linear array. The idea is similar to the triangular array for the QRD proposed by Gentleman and Kung [11]. The difference is that their scheme is a general one without considering any special structure of the data matrix. Accordingly, a full triangular array is needed.

In our approach, due to the consideration of the special Toeplitz-Hankel structure, only a linear array is used and which consists of a boundary cell that generates rotation parameters and $(M-1)$ rotation cells that rotate and up/down-date the input data. The linear array is shown in Fig.1. Once the first row of the matrix R is obtained, the recursion in (23) is then started and the data flow is also shown in Fig.1.

The number of time steps required is now being further reduced to $2(N-M)+(M-1) = 2N-M-1$ (or $2m+n-1$ for a $2m \times n$ Toeplitz-Hankel matrix) based on this linear systolic array implementation.

5 Conclusions

In this paper, we propose a fast algorithm for the QRD of a Toeplitz-Hankel matrix. The computational complexity for the QRD of a $2m \times n$ Toeplitz-Hankel matrix is $10n^2 + 4mn + O(n)$ multiplications, which has an order of magnitude improvement over conventional algorithms. This algorithm can also be implemented onto a linear systolic array. The number of time steps required is further reduced to $2m+n-1$ for the parallel implementation.

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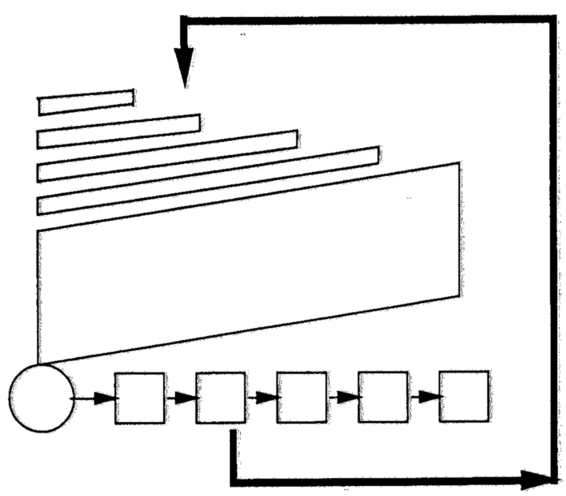

Fig.1 Linear systolic array for fast Toeplitz-Hankel orthogonalization