A GAME THEORETICAL APPROACH FOR IMAGE DENOISING

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ABSTRACT

How to adaptively choose optimal neighborhoods is very important to pixel-domain image denoising algorithms since too many neighborhoods may cause over-smooth artifacts and too few neighborhoods may not be able to efficiently remove the noise. While the Stein's principle is shown to be able to estimate the true mean square error (MSE) for determining the optimal neighborhoods, there exists a trade-off between the accuracy of the estimate and the minimum of the true MSE. In this paper, we study the impact of this trade-off and formulate the image denoising problem as a coalition formation game. In the game, every pixel is treated as a player, who tries to seek partners to form a coalition to achieve better denoising results. By forming a coalition, every player in the coalition can obtain a gain of improving the accuracy of the Stein's estimate while incurring a cost of increasing the minimum of the true MSE. We also propose a heuristically distributed approach for coalition formation. Finally, experimental results show that the proposed game theoretical approach can achieve better performance than the nonlocal method in terms of both PSNR and visual quality.

Index Terms— Image denoising, game theory, coalition formation, Stein's principle.

1. INTRODUCTION

During the processes of being captured, digitized, recorded, and transmitted, an image is usually distorted. The distorted image is visually annoying and make it more difficult to perform tasks such as segmentation, recognition and compression. Therefore, it is very important to reconstruct a good estimate of the image from the corrupted observations.

Many approaches have been proposed in the literature to reconstruct the original image by exploiting the inherently spatial correlation. For examples, Woods and Radewan [1] proposed to estimate the original image from the noisy image by using Kalman filter while Jin [2] proposed to use adaptive Wiener filter. To further exploit the spatial correlation, Buades [3] proposed to average, in a weighted manner, all the pixels in a nonlocal window instead of only involving the locally neighboring pixels. For better reconstruction, nonlinear approaches such as diffusion [4], total variation [5] and fuzzy filtering [6] were also investigated. Besides the pixel-domain approaches, transform-domain approaches such as wavelet shrinkage [7] were also investigated.

Most of the existing schemes focus on how to choose good weights for given neighborhoods to achieve better reconstructions.

However, how to adaptively choose optimal neighborhoods is also very important since too many neighborhoods may cause over-smooth artifacts while too few neighborhoods may not be able to efficiently remove the noise. Due to the absence of the original image, the Stein's principle [8] is used to estimate the true MSE for determining the optimal neighborhoods. Nevertheless, we find that there exists a trade-off between the accuracy of the estimate and the minimum of the true MSE. In this paper, we study the impact of this trade-off and formulate the image denoising problem as a coalition formation game. In the game, every pixel is treated as a player, who tries to seek partners to form a coalition to improve the accuracy of the Stein's estimate while incurring a cost of increasing the minimum of the true MSE. We also propose a heuristic distributed approach for coalition formation. Finally, experimental results show that the proposed game theoretical approach can achieve better performance than the nonlocal method in terms of both PSNR and visual quality. To the best of our knowledge, this paper is the first work to apply game theory for image denoising problems.

The rest of this paper is organized as follows. In Section 2, we describe the system model. Then, we show in details the proposed image denoising games and the proposed heuristic coalition formation approach in Section 3. Finally, we show the experimental results in Section 4 and draw conclusions in Section 5.

2. THE SYSTEM MODEL

In this paper, we consider the problem of restoring images degraded by additive white Gaussian noise. The degraded process can be modelled as

$$I^n(\mathbf{k}) = I(\mathbf{k}) + n(\mathbf{k}), \tag{1}$$

where I is the original image, I^n is the noisy observation of the image, and n is the additive Gaussian noise with zero mean and σ^2 noise variance. The $\mathbf{k} = (k_1, k_2)$ is the coordinate of a pixel. The problem is to find an estimate $\hat{I}$ of the original image based on the noisy observation I^n.

It is well known that the image restoration problem is ill-posed. To reconstruct the original image from the noisy observation, we need to exploit the spatial correlation. In this paper, we focus on the spatially adaptive linear filtering approach. For the pixel located at $\mathbf{k}$, we find the estimate $\hat{I}(\mathbf{k})$ using the weighted average of the spatially neighboring pixels, i.e.,

$$\hat{I}(\mathbf{k}) = \frac{\sum_{\mathbf{l} \in S(\mathbf{k})} w_{\mathbf{k},\mathbf{l}} I^n(\mathbf{l})}{\sum_{\mathbf{l} \in S(\mathbf{k})} w_{\mathbf{k},\mathbf{l}}}, \tag{2}$$
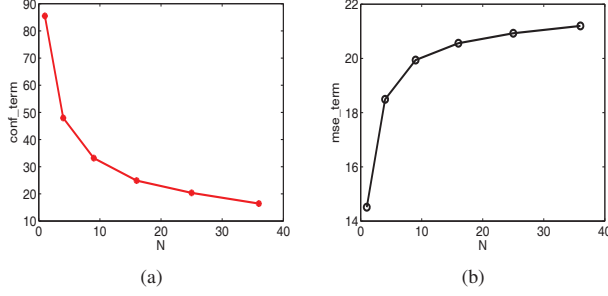
Fig. 1. The trade-off between the confidence term C and the distortion term D: (a) the performance of C with different N; (b) the performance of D with different N.

where $S(\mathbf{k})$ is the candidate set that contains the spatially neighboring pixels for $\mathbf{k}$, and $w_{\mathbf{k},\mathbf{l}}$ is the weight for pixel $I^n(\mathbf{l})$.

In general, the weights $w_{\mathbf{k},\mathbf{l}}$ are determined by the correlation between pixels $I(\mathbf{k})$ and $I(\mathbf{l})$, and should be chosen to minimize the difference between the estimation $\hat{I}(\mathbf{k})$ and the original pixel $I(\mathbf{k})$. However, due to the absence of the original pixel $I(\mathbf{k})$, it is impossible for us to find the optimal weights. One possible approximation is to use the similarity between the neighborhoods around $\mathbf{k}$ and $\mathbf{l}$ [3], which is defined as follows

$$w_{\mathbf{k},\mathbf{l}} = \exp\left\{ -\frac{\sum_{\mathbf{b}\in B}\left[I^n(\mathbf{k}+\mathbf{b}) - I^n(\mathbf{l}+\mathbf{b})\right]^2}{h^2} \right\}, \quad (3)$$

where B is a predefined neighborhood and h is the parameter related to the noise's variance.

Since the weights in (3) are not optimal, the selection of the candidate set $S(\mathbf{k})$ becomes critically important. On one hand, if the size of the candidate set is too small, then the noise may not be efficiently removed. On the other hand, if the size of the candidate set is too large, then the reconstruction may be over-smooth. Moreover, according to (3), we can see that the pixels that are more similar to the target pixel would have larger weights. To prevent the reconstruction from being over-smooth, we will only involve the pixels that have relatively large weights. Let $\Omega(m)$ stand for the subset of $S(\mathbf{k})$ which contains the pixels with the first m largest weights. Then, the reconstruction $\hat{I}(\mathbf{k}, m)$ using $\Omega(m)$ can be written as

$$\hat{I}(\mathbf{k}, m) = \frac{\sum_{\mathbf{l}\in\Omega(m)} w_{\mathbf{k},\mathbf{l}} I^n(\mathbf{l})}{\sum_{\mathbf{l}\in\Omega(m)} w_{\mathbf{k},\mathbf{l}}}. \quad (4)$$

Obviously, the parameter m in (4) should be chosen in such a way that the difference between $\hat{I}(\mathbf{k}, m)$ and $I(\mathbf{k})$ is minimized, i.e., the optimal $m^\star$ can be found by

$$m^\star = \arg\min_m |\hat{I}(\mathbf{k}, m) - I(\mathbf{k})|^2. \quad (5)$$

Algorithm 1 A Heuristic Approach For Coalition Formation

Initialization: let the set of denoised pixel $S_D = \emptyset$ and its complement $\bar{S}_D = \Phi$, let $N_1 = 800$, $N_2 = 21 \times 21$, and $i = 0$.
While $\bar{S}_D \neq \emptyset$

- $i = i + 1$.

- randomly choose $\mathbf{k} \in \bar{S}_D$, let $\Phi^0 = \{\mathbf{k}\}$ and set $j = N_1$.

- While $j > 0$

    - $j = j - 1$.

    - $(\mathbf{l}^\star, m^\star) = \arg\min_{\mathbf{l}\in\bar{S}_D\backslash\Phi^0, 1\le m\le N_2} SURE\left(\Phi^0 \cup \{\mathbf{l}\}, m\right)$.

    - set $\Phi^0 = \Phi^0 \cup \{\mathbf{l}^\star\}$.

    - compute $u(|\Phi^0|) = \frac{g(|\Phi^0|)}{|\Phi^0|} - SURE\left(\Phi^0, m^\star\right)$.

    End

- let $m_i^\star = \arg\max_n u(n)$, $\Phi_i = \Phi^0(1 : m_i^\star)$.

- set $\bar{S}_D = \bar{S}_D \backslash \Phi_i$ and $S_D = S_D \cup \Phi_i$.

- denoise the pixel in Φ_i using (4) with $m = m_i^\star$.

End

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3. GAME THEORETICAL PROBLEM FORMULATION

3.1. Distortion and Confidence Trade-off

Since $I(\mathbf{k})$ is unknown, the optimal $m^\star$ can not be explicitly computed using (5). Fortunately, we can use the Stein's unbiased risk estimate (SURE) [8] to estimate the true mean squared error (MSE) and thus to find the optimal $m^\star$. Suppose that the whole image is partitioned to subsets $\Phi = \{\Phi_1, \Phi_2, ..., \Phi_M\}$, with $m_i^\star$ being the optimal parameter for the subset Φ_i, i.e.,

$$m_i^\star = \arg\min_m \sum_{\mathbf{k}\in\Phi_i} |\hat{I}(\mathbf{k}, m) - I(\mathbf{k})|^2. \quad (6)$$

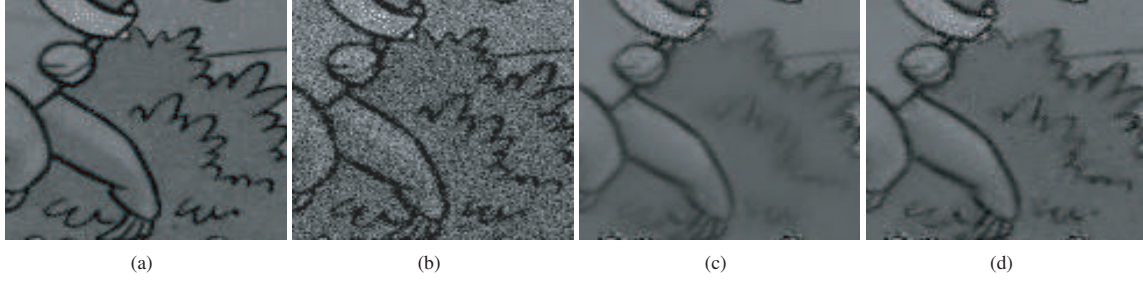Then, the mean square error for the whole image, D, can be computed by

$$D = \frac{1}{|\Phi|} \sum_{i=1}^{M} |\Phi_i| \times mse_i, \quad (7)$$

with

$$mse_i = \frac{1}{|\Phi_i|} \sum_{\mathbf{k}\in\Phi_i} |\hat{I}(\mathbf{k}, m_i^\star) - I(\mathbf{k})|^2, \quad (8)$$

which can be approximated using SURE [8] as follows

$$\begin{aligned} SURE_i &= \frac{1}{|\Phi_i|} \sum_{\mathbf{k}\in\Phi_i} |\hat{I}(\mathbf{k}, m_i^\star) - I^n(\mathbf{k})|^2 \\ &+ \sigma^2\left(\frac{2}{|\Phi_i|} \sum_{\mathbf{k}\in\Phi_i} \frac{\partial \hat{I}(\mathbf{k}, m_i^\star)}{\partial I^n(\mathbf{k})} - 1\right). \end{aligned} \quad (9)$$

| (a) | (b) | (c) | (d) |

Fig. 2. The visual quality comparison for Flinstones with $\sigma = 25$: (a) original image; (b) noisy image; (c) the result generated by the nonlocal method; (d) the result generated by the proposed approach.

To measure the accuracy of the approximation, let us define the confidence term, C, as

$$C = \frac{1}{|\Phi|} \sum_{i=1}^{M} |\Phi_i| \times |mse_i - SURE_i|. \tag{10}$$

According to [8], the confidence term C decreases as $|\Phi_i|$ increases. On the other hand, the distortion term D in (7) increases as $|\Phi_i|$ increases. Therefore, there exists a trade-off between C and D. In Figure 1, we verify this trade-off through experiments by setting $|\Phi_i| = N, \forall i$.

3.2. Utility Function and Solution to the Game

From the previous subsection, we can see that given the partition $\Phi = \{\Phi_1, \Phi_2, ..., \Phi_M\}$, SURE can be used to approximate the true MSE to find the optimal $m^\star$. However, how to find a good partition is not trivial since the number of the partition is not fixed and the size of each partition can vary. Due to the uncertainty of the number of the partition, the traditional segmentation and clustering methods may not work. To study the complex interactions among different pixels and the dynamic partition formation process, we propose to use the coalition formation game.

In this game theoretical formulation, every pixel is treated as a player, who tries to seek partners to form coalitions to achieve better reconstruction. By forming a coalition, every player in the coalition can obtain a gain of reducing the difference between the SURE and the true estimate, i.e., the confidence term in (10), while incurring a cost of increasing the minimum of the MSE. With this idea in mind, we define the utility for a coalition as

$$U(\Phi_i) = -|\Phi_i| \times SURE_i + g(|\Phi_i|), \tag{11}$$

where the first term of the right hand side is the cost, which is an decreasing function of $|\Phi_i|$, while the second term $g(|\Phi_i|)$ is the gain, which should be an increasing function of $|\Phi_i|$. Moreover, with the same $|\Phi_i|$, the gain should decrease as noise variance σ^2 increases. Therefore, we define the gain function as

$$g(|\Phi_i|) = -\lambda \frac{\sigma^2}{|\Phi_i|}, \tag{12}$$

where λ is a parameter.

With the utility function in (11), we can see that as the size of the coalition increases, the members in the coalition can obtain gains from $g(|\Phi_i|)$. However, the gains are limited by the a cost of forming the coalition, which is $-|\Phi_i| \times SURE_i$. The problem now is to find the optimal coalition structures based on the utility function in (11). One possible approach is to use the merge and split rules proposed in [9]. The authors prove that their algorithm will converge to a unique solution with arbitrary merge and split iterations. However, the computation complexity is too high when the size of the player set is large. To make the problem traceable, in this paper, we propose a heuristic approach for coalition formation. The basic idea is to start with a randomly chosen pixel and find the corresponding coalition by selecting the neighborhoods that can give best average utility. The details of the heuristic approach are shown in Algorithm 1.

4. EXPERIMENTAL RESULTS

We evaluate the proposed game theoretical image denoising approach by comparing it with the nonlocal image denoising method [3]. Four 512×512 images: Lena, Barbara, Boat and Flinstones, are tested. The parameter λ in the gain function in (12) is set to be 0.875.

We first evaluate the PSNR comparison versus the noise variance. As show in Figure 5, the proposed method performs much better than the nonlocal method at all tested noise variances. And the gap becomes larger as σ increases. Note that due to page limitation, we only show the results for Barbara and Boat. Similar results are observed for Lena and Flinstones.

The visual quality of the reconstructions are also evaluate. In Figure 2, we show the visual quality comparison for Flinstones. As shown in Fig. 2, (a) is the original patch of Flinstones and (b) is the noisy patch with noise variance $\sigma = 25$. The results generated by the nonlocal method and the proposed approach are shown in (c) and (d) respectively. We can see that the result generated by the nonlocal method is over-smooth. This phenomenon is because the nonlocal method involves too many dis-similar pixels in the averaging process. With the proposed approach, every pixel (player) seeks parters to form coalition to determine the best number of neighborhoods to perform denoising, which can rule out the dis-similar neighborhoods and avoid over-smooth artifacts. Therefore, the detailed can be well-preserved in the proposed ap-
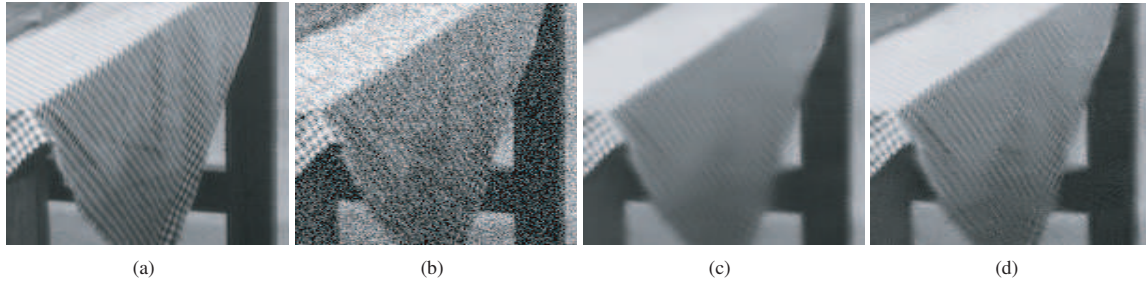
Fig. 3. The visual quality comparison for Barbara with $\sigma = 35$: (a) original image; (b) noisy image; (c) the result generated by the nonlocal method; (d) the result generated by the proposed approach.

Fig. 4. The visual quality comparison for Lena with $\sigma = 45$: (a) original image; (b) noisy image; (c) the result generated by the nonlocal method; (d) the result generated by the proposed approach.
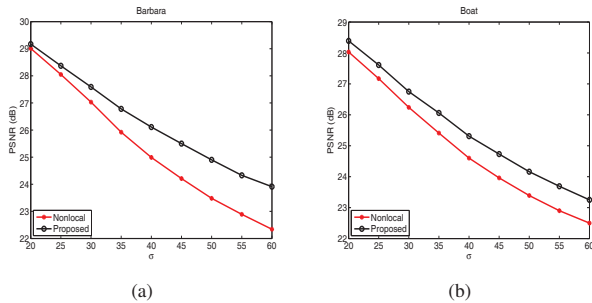
Fig. 5. The PSNR comparison for different images: (a) Barbara; (b) Boat.

proach. Similar results can be observed in Figure 3 and 4 for difference images at different noise variance.

5. CONCLUSION

In this paper, we study the trade-off between the accuracy of the Stein's estimate and the minimum of the true MSE and formulate the image denoising problem as a coalition formation game. With the proposed game, every player (pixel) seek parters to form coalitions to obtain better decision for the optimal neighborhoods and thus lead to better denoising results. We also proposed a heuristic approach for the coalition formation. The experimental results show that compared with nonlocal method, the proposed game theoretical approach can achieve not only better PSNR performance but also better visual quality.

6. REFERENCES

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