An Evolutionary Game-Theoretic Modeling for Heterogeneous Information Diffusion

Xuanyu Cao*, Yan Chen*, Chunxiao Jiang†, K. J. Ray Liu*
*Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742, USA
†Department of Electronic Engineering, Tsinghua University, Beijing, 100084, China

***Abstract*—In this paper, we model and analyze the information diffusion in heterogeneous social networks from an evolutionary game perspective. Users interact with each other according to their individual fitness, which are heterogeneous among different user types. We first study a model where in each social interaction the payoff of a user is independent of the type of the interacted user. In such a case, we derive the information diffusion dynamics of each type of users as well as that of the overall network. The evolutionarily stable states (ESSs) of the dynamics are determined accordingly. Afterwards, we investigate a more general model where in each interaction the payoff of a user depends on the type of the interacted user. We show that the local influence dynamics change much more quickly than the global strategy population dynamics and the former keeps track of the latter throughout the information diffusion process. Based on this observation, the global strategy population dynamics are derived. Finally, simulations are conducted to verify the theoretical results.**

I. Introduction

Social networking is becoming ubiquitous nowadays, which draws great research attention recently. In large-scale social systems, numerous users are exchanging various information every day and the tremendous information flow creates the "Big Data" phenomenon. In the literature, great efforts have been made to understand the information diffusion process over the social networks [1]–[6]. Pinto et al. in [1] predict the future information diffusion by using early popularity data on Youtube, while Weng et al. in [2] take the network and community structure into account to predict successful memes. Several general information diffusion dynamics patterns are identified in [3], and the diffusion cascades are studied in [4]. The great success of some popular social networks such as Facebook and Twitter provides researchers with extensive data to analyze information diffusion, e.g., Wang et al. [5] study the influence of a node on Twitter from the rate of diffusion, and Ilyas et al. [6] consider the rumor spreading on Facebook.

Most of the existing works analyze the information diffusion using data mining/machine learning techniques while completely ignoring the interactions among the rational individuals. However, the overall information diffusion process is determined by the actions of numerous rational individuals and hence it is important to study it from a microeconomics perspective [7]. In this regard, Goyal and Kearns [13] studied the competitive contagions in networks by using game theory while Kampe et al. [14] proposed algorithms to find the initial targets to maximize the future contagions. Additionally, we found in [9] and [10] that the information diffusion process in social networks is actually very similar to the evolution process in ecological systems and thus can be modeled as an evolutionary game [8]. With a graphical evolutionary game approach, we derive the evolutionarily stable states (ESSs) of the diffusion process in [9] and evolutionary dynamics in [10]. One assumption we made before is that the underlying social network is homogeneous, i.e., all users have the same degree of interest and influence in the diffusion process. But, in practice, social systems generally exhibit significant heterogeneity among different users. For example, some may be interested in sport news while some may not; some may be active in a social system while some tends to be silent. This motivates the understanding of real heterogeneous social networks.

In this paper, we model and analyze the information diffusion over heterogeneous social networks by introducing heterogeneous payoff matrices to the evolutionary game theoretic framework. We propose two different models, namely the Type Independent Model and the Type Dependent Model, to characterize the heterogeneity of the social networks to different extent. For both models, we derive the evolutionary dynamics of the information diffusion states. Accordingly, we determine the ESSs of the Type Independent Model and observe that the local dynamics will always keep track of the corresponding global dynamics in the Type Dependent Model. Finally, simulations are conducted to validate our theoretical results.

II. System Model

Consider a social network as a graph with nodes representing users and edges representing relationships between users, e.g., friendship in Facebook. The information diffusion over the network can be modeled as a graphical evolutionary game as follows. The players of the game are the N users. To characterize the heterogeneity of the network, we categorize the users into M types and the payoffs of the interactions between users depend on their types, which will be specified later. The proportion of type-i users in the entire social networks is denoted as $q(i)$. Therefore, $\sum_{i=1}^{M} q(i) = 1$. We assume each user has k neighbors (friends), i.e., a k-regular network. When a piece of information emerges, each user has two possible strategies: forwarding the information (S_f) or not forwarding it (S_n). We denote $p_f(i)$ the proportion of users adopting strategy S_f among all the type-i users. Thus, $p_f = \sum_{i=1}^{M} q(i)p_f(i)$ is the proportion of users adopting strategy S_f among users of all types. The definitions for $p_n(i)$

and p_n are analogous. In heterogeneous social networks, each user knows its own type. But it may or may not know the types of its neighbors since the type of a user is private information, which can only be inferred gradually. For this regard, we present two models for the heterogeneous payoffs, namely the *Type Independent Model* and the *Type Dependent Model*, which are specified as follows.

A. Type Independent Model

In this model, we assume a player's utility does not depend on the type of the interacted user in a social interaction. However, the utility does depend on the player's own type. This can model the case where the type of a user is a private information that others do not know. Specifically, we model the payoff matrix of a type-i user as follows:

$$\begin{array}{cc} & \begin{array}{cc} S_f & \quad S_n \end{array} \\ \begin{array}{c} S_f \\ S_n \end{array} & \begin{pmatrix} u_{ff}(i) & u_{fn}(i) \\ u_{fn}(i) & u_{nn}(i) \end{pmatrix}. \end{array}$$

When a type-i user with strategy S_f is interacting with an user with strategy S_f, its payoff will be $u_{ff}(i)$ regardless of the type of the interacted user. The definitions of $u_{fn}(i)$ and $u_{nn}(i)$ are similar. Here, a symmetric payoff structure is considered as in [9], [10]. In other words, when a type-i user with strategy $S_f(S_n)$ meets an user with strategy $S_n(S_f)$, its payoff will be $u_{fn}(i)$. The value of the payoff matrix depends on both the content of the information and types of the users. For example, if the information is a recent hot topic and forwarding it can attract more popularity from others, then generally $u_{ff}(i)$ is big while $u_{nn}(i)$ is small. And if some users are very interested in that hot topic or they strongly desire to become more popular, then they may have larger $u_{ff}(i)$ and smaller $u_{nn}(i)$ compared to other types of users.

Suppose a type-i user adopting strategy S_f has k neighbors, among which k_f users also adopt strategy S_f. Then, the fitness of this user is defined as:

$$\pi_f(i,k_f) = 1-\alpha+\alpha[k_f u_{ff}(i)+(k-k_f)u_{fn}(i)], \quad (1)$$

where α is the selection strength and the baseline fitness is normalized to 1. We assume in this paper that α is very small as in [9]–[12], i.e., weak selection. Similarly, if the user adopts strategy S_n, its fitness will be:

$$\pi_n(i,k_f) = 1-\alpha+\alpha[k_f u_{fn}(i)+(k-k_f)u_{nn}(i)]. \quad (2)$$

The (1) and (2) clearly indicate that a user's fitness is influenced by its interactions with its neighbors.

B. Type Dependent Model

Through repeated interactions, a user may somehow manage to know the types of its neighbors. To model such a scenario, we propose the Type Dependent Model specified as follows.

Consider a type-i user A, which is interacting with one of its type-j neighbor B. Since A knows the type of B, the payoff of A in this interaction should depend on the type of B. Specifically, if A, B both adopt strategy S_f, then the payoff of

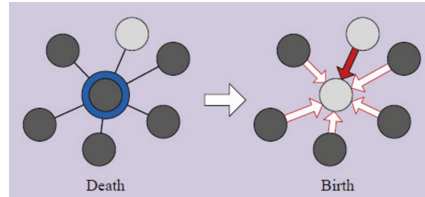

Fig. 1. DB update rule. The central user is selected to update its strategy and it chooses the gray neighbor's strategy for its new strategy.

A is $u_{ff}(i,j)$. If A, B adopt strategy S_f and S_n respectively, then the payoff of A is $u_{fn}(i,j)$. Similarly, we can define $u_{nf}(i,j)$ and $u_{nn}(i,j)$. From these notations, we can see that in Type Dependent Model, the payoff of a user depends on not only its own type but also the type of the interacted user, i.e., to identify a payoff we need a pair of type arguments (i,j).

As for the fitness, similar to (1) and (2), the fitness of a user with strategy S_f or S_n is given by:

$$\pi_f(i) = 1-\alpha+\alpha\sum_{j=1}^{M}[k_f(j)u_{ff}(i,j)+k_n(j)u_{fn}(i,j)], \quad (3)$$

$$\pi_n(i) = 1-\alpha+\alpha\sum_{j=1}^{M}[k_f(j)u_{nf}(i,j)+k_n(j)u_{nn}(i,j)], \quad (4)$$

where $k_f(j)$ ($k_n(j)$) denotes the number of type-j users with strategy S_f (S_n) among all the neighbors.

With the definition of fitness in both models, we could specify the strategy update rule of the users. In this paper, we adopt the death-birth (DB) update rule. At each time slot, one user is randomly and uniformly selected to update its strategy. Then, the chosen user will adopt one of its neighbor's strategy with probability proportional to fitness, as shown in Fig. 1.

III. Analysis for the Type Independent Model

In this section, we derive the evolutionary dynamics of the network states $p_f(i)$ and accordingly determine the ESSs under the Type Independent Model.

In a certain time slot, suppose a type-i user with strategy S_f (in the following, we will call this user as the center user) is chosen to update its strategy. Among its k neighbors, there are k_f users adopting strategy S_f and $(k-k_f)$ users adopting strategy S_n. Thus, the fitness $\pi_f(i,k_f)$ of the center user is given in (1). If the center user changes its strategy to S_n, its fitness will become $\pi_n(i,k_f)$ (2). In DB update rule, the center user updates its strategy according to the fitness of its neighbors. Since different types of users generally have different payoff matrices, the center user only want to learn strategies from its neighbors of the same type, i.e., type i. However, in the Type Independent Model, the center user does not know the types of its neighbors. So, the center user has to treat all of its neighbors as type i. Thus, from the perspective of the center user, a neighbor adopting strategy S_f (S_n) has fitness $\pi_f(i,k_f)$ ($\pi_n(i,k_f)$). According to the DB update rule, the probability that the center user changes its strategy from

S_f to S_n is given by:

$$P_{f\to n}(i,k_f)=\frac{(k-k_f)\pi_n(i,k_f)}{k_f\pi_f(i,k_f)+(k-k_f)\pi_n(i,k_f)}. \quad (5)$$

Substituting (1) and (2) into (5) yields:

$$\begin{aligned}&P_{f\to n}(i,k_f)\\&=\frac{k-k_f}{k}+\alpha(k-k_f)\left[\frac{k_f^2}{k^2}\Delta(i)+\frac{k_f}{k}\Delta_n(i)\right]+O(\alpha^2),\end{aligned} \quad (6)$$

where $\Delta(i):=2u_{fn}(i)-u_{ff}(i)-u_{nn}(i)$, $\Delta_n(i):=u_{nn}(i)-u_{fn}(i)$. Because α is very small due to weak selection, we will omit the $O(\alpha^2)$ term in the following. Since the proportion of users with strategy S_f is p_f over the entire network, each neighbor has probability p_f of adopting strategy S_f. Thus k_f is a binomial distributed random variable with probability mass function: $\theta(k,k_f)=\binom{k}{k_f}p_f^{k_f}(1-p_f)^{k-k_f}$. Hence, taking expectation of (6) gives:

$$\begin{aligned}\mathbb{E}[P_{f\to n}(i,k_f)]=1-p_f+\frac{\alpha}{k^2}\Delta(i)[&-k(k-1)(k-2)p_f^3\\&+(k^3-4k^2+3k)p_f^2+(k^2-k)p_f]\\&+\frac{\alpha}{k}\Delta_n(i)\left[-k(k-1)p_f^2+k(k-1)p_f\right].\end{aligned} \quad (7)$$

In each round of DB update, one of the N users will be selected to update its strategy randomly. The proportion of type-i users with strategy S_f among all the users is $p_f(i)q(i)$. Thus, we have:

$$\Pr\left(\delta p_f(i)=-\frac{1}{Nq(i)}\right)=p_f(i)q(i)\mathbb{E}[P_{f\to n}(i,k_f)], \quad (8)$$

where δ denotes increment. With a similar argument as above, one can compute the probability that a type-i user changes its strategy from S_n to S_f as follows:

$$\Pr\left(\delta p_f(i)=\frac{1}{Nq(i)}\right)=p_n(i)q(i)(1-\mathbb{E}[P_{f\to n}(i,k_f)]). \quad (9)$$

Combining (7), (8) and (9), we can derive the dynamic of $p_f(i)$ as follows:

$$\begin{aligned}&\dot{p}_f(i)\\&=\frac{1}{Nq(i)}\left[\Pr\left(\delta p_f(i)=\frac{1}{Nq(i)}\right)-\Pr\left(\delta p_f(i)=-\frac{1}{Nq(i)}\right)\right]\\&=\frac{1}{N}p_f-\frac{1}{N}p_f(i)\\&+\frac{\alpha(k-1)}{Nk}p_f(p_f-1)[\Delta(i)((k-2)p_f+1)+k\Delta_n(i)].\end{aligned} \quad (10)$$

Hence, the dynamic of p_f is given by:

$$\begin{aligned}\dot{p}_f&=\sum_{i=1}^{M}q(i)\dot{p}_f(i)\\&=\frac{\alpha(k-1)}{Nk}p_f(p_f-1)[(k-2)\overline{\Delta}p_f+\overline{\Delta}+k\overline{\Delta}_n],\end{aligned} \quad (11)$$

where $\overline{\Delta}:=\sum_{i=1}^{M}q(i)\Delta(i)$ and $\overline{\Delta}_n:=\sum_{i=1}^{M}q(i)\Delta_n(i)$. The evolutionary dynamics and their ESSs are summarized in the following theorem. The proof is omitted due to space limitation.

Theorem 1: In the Type Independent Model, the evolutionary dynamics of the network states $p_f(i), 1\le i\le M$ are given by (10) while the evolutionary dynamic of p_f is given by (11). The corresponding ESSs of the dynamical system can be summarized as follows:

$$p_f^*=\begin{cases}0, & \text{if } \overline{u}_{nn}>\overline{u}_{fn},\\ 1, & \text{if } \overline{u}_{ff}>\overline{u}_{fn},\\ \dfrac{(k-2)\overline{u}_{fn}-(k-1)\overline{u}_{nn}+\overline{u}_{ff}}{(k-2)(2\overline{u}_{fn}-\overline{u}_{nn}-\overline{u}_{ff})}, & \text{if } \max\{\overline{u}_{ff},\overline{u}_{nn}\}<\overline{u}_{fn},\end{cases} \quad (12)$$

$$\begin{aligned}p_f^*(i)=p_f^*-\frac{\alpha(k-1)}{k}p_f^*(1-p_f^*)[((k-2)p_f^*+1)\\ \cdot(2u_{fn}(i)-u_{nn}(i)-u_{ff}(i))+k(u_{nn}(i)-u_{fn}(i))],\end{aligned} \quad (13)$$

where $\overline{u}_{ff}=\sum_{i=1}^{M}q(i)u_{ff}(i)$ and $\overline{u}_{fn},\overline{u}_{nn}$ are similarly defined.

IV. Analysis for the Type Dependent Model

In this section, we first discuss about the network states in the Type Dependent Model and then derive their evolutionary dynamics.

A. Network States

In the Type Dependent Model, since a user's type and strategy affect its neighbors' utilities, they will also influence the neighbors' strategies. Thus, the edge information is also required to fully characterize the network state. Specifically, we define network edge states as $p_{ff}(i,j), p_{fn}(i,j), p_{nn}(i,j)$, where $p_{ff}(i,j)$ ($p_{nn}(i,j)$) denotes the proportion of edges connecting a type-i user with strategy S_f (S_n) and a type-j user with strategy S_f (S_n), and $p_{fn}(i,j)$ denotes the proportion of edges connecting a type-i user with strategy S_f and a type-j user with strategy S_n. Moreover, we denote $p_{f|f}(i,j)$ the percentage of type-i neighbors adopting strategy S_f, given a center type-j user using strategy S_f. Similarly, we can define $p_{f|n}(i,j), p_{n|f}(i,j), p_{n|n}(i,j)$. In summary, we have *population states* (e.g. $p_f(i)$), *relationship states* (e.g. $p_{ff}(i,j)$) and *influence states* (e.g. $p_{f|f}(i,j)$) as our network states. Since these states are dependent with each other, we only need a subset of them to characterize the entire network state. For example, we can use $p_f(i), 1\le i\le M$ and $p_{ff}(i,j), 1\le i,j\le M$ to compute all the other states.

B. Analysis

Consider a type-i user using strategy S_f (the center user), who is chosen to update its strategy. Then, its fitness is given in (3). Rigorously speaking, $k_f(j)$ and $k_n(j)$ in (3) are random variables with expection $kq(j)p_{f|f}(j,i)$ and $kq(j)p_{n|f}(j,i)$ respectively. Since in real world social networks, k is relatively large (more than 100 for typical online social networks such as Facebook) and a small number of types (i.e., M) is enough to capture the user behaviors, we substitute $k_f(j), k_n(j)$ by their expectations for ease of analysis in the following. Hence, (3) becomes:

$$\pi_f(i)=1-\alpha+\alpha k\sum_{j=1}^{M}q(j)[p_{f|f}(j,i)u_{ff}(i,j)+p_{n|f}(j,i)u_{fn}(i,j)]. \quad (14)$$

Similarly, if a type-i user is adopting strategy S_n, its fitness in (4) can be approximated as:

$$\pi_n(i) = 1-\alpha+\alpha k\sum_{j=1}^{M} q(j)[p_{f|n}(j,i)u_{nf}(i,j)+p_{n|n}(j,i)u_{nn}(i,j)]. \tag{15}$$

The center user should only update its strategy according to its type-i neighbors since other types of neighbors have different payoff matrices and learning strategies from them is meaningless. Furthermore, on average, there are $kp_{f|f}(i,i)$ type-i neighbors using strategy S_f and $kp_{n|f}(i,i)$ type-i neighbors using strategy S_n. Thereby, according to the DB update rule, the probability that the center user will update its strategy to be S_n is:

$$\mathbb{P}_{f\to n}(i) = \frac{\pi_n(i)p_{n|f}(i,i)}{\pi_f(i)p_{f|f}(i,i)+\pi_n(i)p_{n|f}(i,i)}. \tag{16}$$

Then, following a procedure similar to that of the Type Independent Model, we can prove the following two theorems.

Theorem 2: In the Type Dependent Model, the population dynamics $p_f(\cdot)$ are given by:

$$\begin{aligned}\dot{p}_f(i) = {} & \frac{\alpha k}{N}p_f(i)p_{n|f}(i,i)(p_{n|n}(i,i)+p_{f|f}(i,i))\\ & \times\sum_{j=1}^{M} q(j)[p_{f|f}(j,i)u_{ff}(i,j)+p_{n|f}(j,i)u_{fn}(i,j)\\ & - p_{f|n}(j,i)u_{nf}(i,j)-p_{n|n}(j,i)u_{nn}(i,j)].\end{aligned} \tag{17}$$

while the relationship dynamics $p_{ff}(\cdot,\cdot)$ are given by ($l \neq i$):

$$\begin{aligned}&\dot{p}_{ff}(i,l)\\ &= \frac{2}{N}q(i)q(l)p_f(i)(1-p_{f|f}(i,i))\left[\frac{p_f(l)}{p_n(i)}(1-p_{f|f}(i,l))-p_{f|f}(l,i)\right]\\ &+\frac{2}{N}q(i)q(l)p_f(l)(1-p_{f|f}(l,l))\left[\frac{p_f(i)}{p_n(l)}(1-p_{f|f}(l,i))-p_{f|f}(i,l)\right],\end{aligned} \tag{18}$$

and

$$\dot{p}_{ff}(i,i) = \frac{2}{Np_n(i)}q^2(i)p_f(i)(1-p_{f|f}(i,i))(p_f(i)-p_{f|f}(i,i)). \tag{19}$$

Theorem 3: Under the Type Dependent Model, in a small time scale where the population states $p_f(\cdot)$ remain constant, the influence dynamics $p_{f|f}(\cdot,\cdot)$ are given by:

$$\dot{p}_{f|f}(l,i) = \frac{1}{N}(p_f(l)-p_{f|f}(l,i))\left[\frac{1-p_{f|f}(i,i)}{p_n(i)}+\frac{1-p_{f|f}(l,l)}{p_n(l)}\right]. \tag{20}$$

From Theorem 3, we observe that the influence states $p_{f|f}(l,i)$ will always keep track of the corresponding population states $p_f(l)$. Hence, we can make the approximation that $p_{f|f}(l,i) = p_f(l), \forall l, i$. Thus, the population dynamics in (17) can be simplified as in the following corollary.

Corollary 1: In the Type Dependent Model, the population dynamics $p_f(i)$ for each type $i = 1, 2, ...M$ are given by:

$$\begin{aligned}\dot{p}_f(i) = {} & \frac{\alpha k}{N}p_f(i)p_n(i)\sum_{j=1}^{M} q(j)[p_f(j)(u_{ff}(i,j)-u_{nf}(i,j))\\ & +p_n(j)(u_{fn}(i,j)-u_{nn}(i,j))].\end{aligned} \tag{21}$$

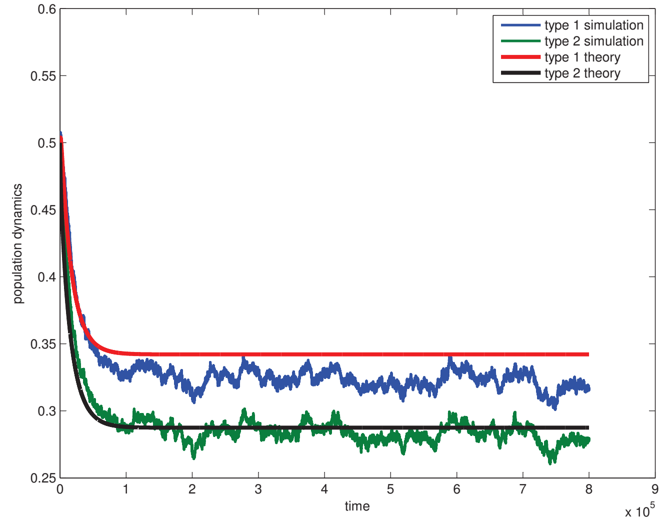

Fig. 2. Simulation results for population dynamics

V. Simulations

In this section, we show simulation results to validate our theoretical results. We focus on the Type Independent Model here, and set the parameters as follows: $N = 1000, k = 20, \alpha = 0.05, u_{ff}(1) = 0.4, u_{ff}(2) = 0.2, u_{fn}(1) = 0.6, u_{fn}(2) = 0.4, u_{nn}(1) = 0.3, u_{nn}(2) = 0.5$. According to Theorem 1, the theoretical ESS can be calculated as $p_f^*(1) = 0.3421, p_f^*(2) = 0.2875$. The theoretical dynamics $p_f(1)$ and $p_f(2)$ can also be computed by (10). On the other hand, we carry out computer simulations based on the model. The result is reported in Fig. 2, where the simulation result is averaged over 100 independent runs. We observe that our theoretical result matches well with the simulation result.

To see the advantage of our heterogeneous modeling approach, we use the homogeneous model in [9] to analyze the heterogeneous setup here. The payoff matrix of each type is set to be the average over all types. Then, the theoretical ESS of p_f can be calculated as 0.3148. Since the network is modeled as homogeneous, the theoretical ESSs of $p_f(1)$ and $p_f(2)$ are both 0.3148. In this case, compared to the simulation result, the average relative ESS error (ESS error over the simulated ESS) of the two types is 6.83% while that in the heterogeneous model is 3.54%, which shows the advantage and necessity of using heterogeneous model.

VI. Conclusion

In this paper, we study the information diffusion process in social networks from an evolutionary game perspective. We propose two game-theoretic models to analyze the diffusion process. Evolutionary dynamics are derived and the ESSs are determined accordingly. Simulations are carried out to confirm the theoretical results. We find that when the underlying social network is heterogeneous, the heterogeneous model gives more accurate theoretical ESSs than the homogeneous model does. For future work, we would like to test our model and theory by using real-world datasets.

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